APPROXIMATION PROPERTIES OF SOME OPERATORS AND THEIR q-ANALOGUES

THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE OF

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IN

MATHEMATICS

By

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Preface

The present thesis entitled *'Approximation Properties of Some Operators and Their q-Analogues'* contains the research work done by me under the constant supervision of Prof. Mursaleen, Department of Mathematics, Aligarh Muslim University, Aligarh. In the present work, we study the approximation of functions by simple and more easily calculated functions i.e., by positive linear operators which play an important role in the theory of approximation. The approximation of functions by positive linear operators is a significant research area in mathematical analysis with key relevance to studies of computer-aided geometric design, numerical analysis, and solution of differential equations. At the last quarter of 20th century, q-calculus appeared as a connection between mathematics and physics. q-calculus is a generalization of many subjects, such as hypergeometric series, complex analysis, and particle physics.

The material in this thesis is divided into seven chapters and each chapter is further divided into sections and subsections as required. The definitions, examples, remarks, theorems etc. have been specified with the single decimal numbers. The first figure denotes the number of the chapter and second represents the position of occurrence of definitions, examples, remarks, theorems, proposition, corollary, etc. For instance, Theorem 4.2 shows the theorem is at the second position in the fourth chapter.

In the first chapter we concisely present preliminary notions and auxiliary results that will be used throughout the thesis. Concretely we give some basic definitions and certain elementary properties of our main tools in providing quantitative estimates concerning positive linear operators, modulus of continuity ω and the moduli of smoothness ω_2 and Peetre's K-functional. We also have given a brief introduction of Hyers-Ulam stability of functional equations but about the operators we discuss in chapter 2. Moreover, basics of quantum calculus, e.g., q-calculus and (p, q)-calculus is also given in details.

The second chapter is dedicated to study of Hyers-Ulam stability of operators. In this chapter, we have shown that the operators from approximation theory, e.g., Bernstein-Stancu, Kantorovich-Stancu etc are stable in Hyers-Ulam sense. Furthemore, we have also established that some of the operators are not stable in the sense of Hyers-Ulam.

The third chapter represents the approximation of functions by Chlodowsky variant Szász operators which involves Brenke polynomials. This can be taken as a generalization of Szász operators. Korovkin's type approximation and local approximation results are studied. Also we show the convergence of r-th order derivative and given an example in support of this result. At last, some convergence properties of these operators in weighted spaces with weighted norm are discussed.

In the fourth chapter, we deals with the approximation properties of q-Stancu-Beta operators. We calculate the rate of approximation and point-wise estimates as well as Voronovskaja type asymptotic formula. We also give some results (e.g global covergence) for q-Stancu-Beta operators preserving x^2 which is a King type modification. These results can be taken as in the continuation of a paper by Q.B. Cai entitled *"Approximation properties of the modification of q-Stancu-Beta operators which preserve* x^2*,"* Journal of inequalities and applications 2014, 2014:505.

The fifth chapter is about to study the approximation properties of the Kantorovich variant of q-Bernstein operators with shifted knots. We prove the basic convergence of the introduced operators and also obtain the rate of convergence in terms of modulus of continuity. Further, we study the local approximation property and Voronovskaja type theorem for the said operators. With the help of the Matlab we show comparisons and some illustrative graphics for the convergence of operators to a function.

In the last two chapters, we have introduced some new generalization of q-operators with the help of (p, q)-calculus which is quite a new idea and not done by any researcher yet. In these chapters we constructed (p, q)-analogue of Bersntein operators, Bernstein-stancu operators, Bernstein-Kantorovich operators and Szász-Kantorovich operators. We have also given some remark how our operators is different from q-operators and it is a consequence of q-operators. We establish some convergence results and local property as well as Voronovskaja type approximation theorem.

Finally at the end, a bibliography is given which by no means is exhaustive one but lists only those books and papers which have been referred in the present thesis.

Chapter 1

Preliminaries and auxiliary results

1.1 Introduction

The *approximation theory* is a very important area of mathematical analysis, which, at its core, is concerned with the approximation of functions by simple and more easily calculated functions. The key moment in the development of approximation theory was in 1885 when Karl Weierstrass [109] presented the first proof of his (fundamental) theorem on approximation by algebraic or trigonometric polynomials, which asserts that *for any continuous function f on the finite interval $[a, b]$, there exists a sequence of polynomials which converges uniformly to f on $[a, b]$*. The Weierstrass approximation theorem differs from Taylor's theorem, which states that a function with sufficiently many derivatives can be approximated locally by its Taylor polynomial. The Weierstrass approximation theorem applies to a continuous function, which need not be differentiable; and states that there is a global polynomial approximation of the function on the whole interval $[a, b]$. We can mention here that some of the great mathematicians they relate their names to this most celebrated theorem: Carl Runge (1885), Henri Lebesgue (1908), Charles de la Vallée-Poussin (1908), Lipot Fejér (1916) and, of course, Sergej N. Bernstein (1912). In 1912, S.N. Bernstein [10] gave a simple, short and most elegant proof of Weiertrass theorem, constructing, by probabilistic methods, a sequence of polynomials that converges uniformly to the function to be approximated. Thus were introduced the (now) very well-known Bernstein operators (the applications that associate the functions to be approximated with the polynomials that approximate the function):

$$B_n(f; x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \tag{1.1}$$

for any $f \in C[0, 1]$, $x \in [0, 1]$. These operators belong to the class of positive linear operators.

The importance of these remarkable operators could not have been anticipated in the first half of last century. In the '50s, the theory of approximation of functions by positive linear operators developed a lot, when three mathematicians in three consecutive years T Popoviciu [91] in 1951, H. Bohman [11] in 1952 and P.P. Korovkin [60] in 1953, discovered independently, a simple and easy proof for the convergence of a sequnece of positive linear operators towads the identity operator. This criterion says that *the necessary and sufficient condition for the uniform convergence of the sequence* (L_n) *of positive linear operators to the continuous function* f *on the compact interval* $[a, b]$*, is the uniform convergence of the sequence* $L_n e_k$ *to* e_k *for* $e_k(x) = x^k$, $k = 0, 1, 2$. If the domain of definition of f is unbounded (for example $[0, \infty)$), then the result remains valid only for the continuous functions having a finite limit at infinity. In this case, the test fuctions, x^k, $k = 0, 1, 2$ are replaced by other three functions ($e^{-kx}, k = 0, 1, 2$ are an example). The classical result of approximation theory is mostly known under the name of *Bohman-Korovkin theorem,* because T. Popoviciu's contribution in [91] remained unknown for a long period of time.

1.2 Positive linear operators

1.2.1 Definition and properties of positive linear operators

Let M be a non empty set and let

$$\mathcal{F}(M, \mathbb{R}) = \{f : M \to \mathbb{R}\},$$

be the linear space over $\mathbb{R}$ of real functions defined on M, endowed with the usual operations of addition and scalar multiplication.

In the following, we denote by X a linear subspace of $\mathcal{F}(M, \mathbb{R})$ and by Y a linear subspace of $\mathcal{F}(N, \mathbb{R})$, where M and N are nonempty sets.

Definition 1.1. *Let* X, Y *be two linear spaces of real functions. The mapping* $L : X \to Y$ *is called a linear operator if*

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g), \text{ for every } f, g \in X, \ \alpha, \beta \in \mathbb{R},$$

and is positive, if

$$L(f) \geq 0, \text{ for every } f \in X \text{ with the property } f \geq 0.$$

Remark 1.2. *In order to highlight the argument of the function $Lf \in Y$ we use the notation $L(f;x)$ but also in some rare cases $(Lf)(x)$.*

Proposition 1.3. *(i) A positive linear operator is monotone.*

(ii) If L is a positive linear operator, then for every $f \in X$ we have $|Lf| \leq L(|f|)$.

Proposition 1.4. *(Hölder inequality for positive linear operators). Let $L : X \to Y$ be a positive linear operator and let $p, q > 1$ be real numbers such that $1/p + 1/q = 1$. Then*

$$L(|f \cdot g|) \leq (L(|f|^p))^{\frac{1}{p}} \cdot (L(|g|^q))^{\frac{1}{q}}, \text{ for every } f, g \in X.$$

Remark 1.5. *An important particular case is the Cauchy-Schwarz inequality for positive linear operators, which is obtained from Hölder inequality for $p = q = 2$, by using Proposition 1.3 (ii):*

$$|L(f \cdot g;x)| \leq \sqrt{L(f^2;x)} \cdot \sqrt{L(g^2;x)}.$$

Definition 1.6. *Let $L : X \to Y$, where $X \subseteq Y$ be two normed linear spaces of real functions. To each operator L we can assign a non-negative number $\|L\|$ defined by*

$$\|L\| := \sup_{\substack{f \in X \\ \|f\|=1}} \|Lf\| = \sup_{\substack{f \in X \\ 0<\|f\|\leq 1}} \|Lf\|.$$

By convention, if X is the zero linear space, any operator L which maps X to Y must be the zero operator and is assigned the zero norm.

It can be verified that $\|.\|$ satisfies all the properties of a norm and hence is called *the operator norm.*

Choosing $X = Y = C[a,b]$ the following can be stated regarding the continuity and the operator norm:

Proposition 1.7. *If $L : C[a,b] \to C[a,b]$ is a positive linear operator. Then L is continuous and has the norm $\|L\| = \|Le_0\|$.*

1.2.2 Examples of positive linear operators

Example 1.8. *(Bernstein operators): For a positive integer $n \geq 1$ and for a function f defined on $[0,1]$, the Bernstein operators $B_n : C[0,1] \to C[0,1]$ are defined by*

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1].$$

Each $B_n(f)$ is a polynomial of degree not greater than n. They were introduced by S.N. Bernstein [10] in 1912 to give the first constructive proof of the Weierstrass approximation theorem (algebraic version) ([109]).

Actually, we have that:

Theorem 1.9. *For every $f \in C[0,1]$,*

$$\lim_{n\to\infty} B_n(f) = f \ \text{uniformly on}[0,1].$$

Note that Theorem 1.9 furnishes a constructive proof of the Weierstrass approximation theorem [109] which we state below. (For a survey on many other alternative proofs of Weierstrass theorem, we refer, e.g., to [86, 87].)

Theorem 1.10. *For every $f \in C([0,1])$, there exists a sequence of algebraic polynomials that uniformly converges to f on $[0,1]$.*

Example 1.11. *(Bernstein-Stancu operators): Let $0 \le \alpha \le \beta$ be real numbers. For $n \ge 1$, the relation*

$$P_n^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k+\alpha}{n+\beta}\right),$$

defines the Bernstein-Stancu operators $P_n^{(\alpha,\beta)} : C[0,1] \to C[0,1]$, introduced by D.D. Stancu [99] in 1969.

Example 1.12. *(Bernstein-Kantorovich operators): The Bernstein polynomials are not suitable to approximate Lebesgue integrable functions, or in other words, general discontinuous functions. Since the space $C[0,1]$ is dense in $L^p[0,1]$ with respect to the natural norm $\|f\|_p := \left(\int_0^1 |f(t)|^p dt\right)^{1/p}$ $(f \in L^p[0,1])$. So, by replacing $f\,(k/n)$ in the definition of Bernstein polynomial by an integral mean of $f(x)$ over a small interval around k/n, we may obtain better results. To approximate Lebesgue integrable functions on the interval $[0,1]$, Kantorovich [56] introduced modified Bernstein polynomials as*

$$K_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt,$$

where

$$p_{n,k}(x) = \binom{n}{k} x^n (1-x)^{n-k}, \ \ 0 \le x \le 1.$$

Each $K_n(f;x)$ is a polynomial of degree not greater than n and K_n is a positive linear operator from $L^p[0,1]$ (and, in particulr from $C[0,1]$) into $C[0,1]$.

For a noncompact interval $I \subseteq \mathbb{R}$, let $\mathcal{D} \subset C(I)$ be a linear space of continuous real functions defined on I. In the following, we give some examples of positive linear operators defined on such a subspace, which will be mentioned for every particular case.

Example 1.13. *(Szász-Mirakjan operators): For* $I = [0,\infty)$, *the operators* $S_n : \mathcal{D} \to C(I)$ *defined by*

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \quad x \in [0,\infty),\ n \geq 1,$$

are called Szász-Mirakjan operators. They were introduced by G. Mirakjan [75] in 1941 (some authors spell this name: Mirakyan) and were studied by J. Favard [29] in 1944 and by O. Szász [104] in 1950. The domain of definition of S_n *is the set of all functions* $f(x) = \mathcal{O}(e^{\alpha x \ln x})$, $\alpha > 0$, *this fact being proved by T. Hermann [43].*

Example 1.14. *(Baskakov operators): For* $I = [0,\infty)$, *the operators* $V_n : \mathcal{D} \to C(I)$ *defined by*

$$V_n(f;x) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k}{n}\right), \quad x \geq 0,\ n \geq 1,$$

are called Baskakov operators and were introduced by V.A. Baskakov [9] in 1957. The domain of definition $\mathcal{D}$, *is the set of all functions* f *which have the growth* $f(x) = \mathcal{O}(e^{\alpha x})$, $\alpha > 0$, *this fact being proved by T. Hermann [43].*

Example 1.15. *(Durrmeyer operators): To approximate Lebesgue integrable functions on the interval* $[0,1]$ *Durrmeyer introduced the integral modification of the well known Bernstein polynomials. In 1981 Derriennic [22] first studied these operators in details. The Durrmeyer operators* D_n *are defined as*

$$D_n(f;x) = (n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt, \quad x \in [0,1],$$

where the Bernstein basis function is defined by $p_{n,k} = \binom{n}{k}x^k(1-x)^{n-k}$.

Example 1.16. *(Beta operators): Stancu [100] introduced Beta operators* L_n *of the second kind in order to approximate the Lebesgue integrable functions on the interval* $(0,\infty)$ *as*

$$L_n(f;x) = \frac{1}{B(nx, x+1)}\int_0^{\infty}\frac{t^{nx-1}}{(1+t)^{nx+n+1}}f(t)dt.$$

Obviously the operators L_n *are positive linear operators on the space of locally integrable functions on* $(0,\infty)$ *of polynomial growth as* $t \to \infty$, *provided that n is sufficiently large.*

Example 1.17. *(Bernstein-Chlodowsky operators): For* $I = [0,\infty)$, *the operators* $C_n : \mathcal{D} \to C(I)$ *defined by*

$$C_n(f;x) = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{\beta_n}\right)^k\left(1-\frac{x}{\beta_n}\right)^{n-k}f\left(\frac{k}{n}\beta_n\right),$$

for $0 \leq x \leq \beta_n$ *and* $C_n(f;x) = f(x)$, *for* $x > \beta_n$, *where* $(\beta_n)_{n\in\mathbb{N}}$ *is a sequence of positive real numbers having the properties*

$$\lim_{n\to\infty}\beta_n = \infty \text{ and } \lim_{n\to\infty}\frac{\beta_n}{n} = 0,$$

are called Bernstein-Chlodowsky operators and were introduced by I. Chlodowsky [20] in 1937.

For more examples, one can refer to [45].

1.3 Approximation of functions by positive linear operators

1.3.1 Different types of moduli of continuity

The first modulus of continuity (smoothness) has a long history. It appeard already in 1911 in the Ph.D. thesis of D. Jackson [50], the work that laid the basis for what is known today as *Quantitative Approximation Theory.*

In 1987, Ditzian and Totik introduced what they call a "natural modulus of smoothness" which is considered to be a "better tool to deal with the rate of best approximation, inverse theorems and embedding theorems" (see [26], p. 1-4).

The Ditzian-Totik modulus of smoothness is given by

$$\omega_{\varphi}^{r}(f,\delta)_{p}=\sup_{0<h\leq\delta}\|\Delta_{h\varphi}^{r}f\|_{L_p} \tag{1.2}$$

where the function $\varphi(x)$ and the interval in question are related to the problem at hand and $\Delta_{h\varphi}^{r}f$ denotes the r-th order of forward difference of f with step lenght $h\varphi$.

Remark 1.18. *A vital feature of (1.2) is that the increment $h\varphi(x)$ varies with x. For $\varphi(x)\equiv 1$, (1.2) is reduced to the classical modulus.*

The main tools to measure the degree of convergence of positive linear operators towards the identity operator are the moduli of smoothness of first and second order. For $f\in C[a,b]$ and $\delta\geq 0$, we have

$$\omega_1(f,\delta):=\sup\{|f(x+h)-f(x)|:\ x,x+h\in[a,b],\ 0\leq h\leq\delta\}$$
$$\omega_2(f,\delta):=\sup\{|f(x+h)-2f(x)+f(x-h)|:\ x,x\pm h\in[a,b],\ 0\leq h\leq\delta\},$$

or, we can write it as

$$\omega_1(f,\delta):=\sup_{0<h\leq\sqrt{\delta}}\sup_{x\in[a,b]}|f(x+h)-f(x)|,\qquad \text{or} \tag{1.3}$$

$$\omega(f,\delta) := \sup_{\substack{x,y\in[a,b]\\ |x-y|\le\delta}} |f(x)-f(y)|,$$

$$\omega_2(f,\delta) := \sup_{0<h\le\sqrt{\delta}} \sup_{x\in[a,b]} |f(x+h)-2f(x)+f(x-h)|. \tag{1.4}$$

Most of the error estimates in this work are given in terms of the two moduli of smoothness, the Ditzian-Totik second order modulus denoted by $\omega_\varphi^2(f,\cdot)$ and sometimes by $\omega_2^\varphi(f,\cdot)$.

$\omega_1 = \omega$ inherits its name from the first part of the following property:

Proposition 1.19. *Let $f \in C[a,b]$ and $\delta > 0$.*

(i) *If f is uniformly continuous on (a,b), then it is necessary and sufficient that $\lim_{\delta\to 0}\omega(f,\delta) = 0$.*

(ii) *For every $\delta > 0$ we have*

$$|f(y)-f(x)| \le \left(1+\frac{|y-x|}{\delta}\right)\omega(f,\delta), \tag{1.5}$$

and

$$|f(y)-f(x)| \le \left(1+\frac{(y-x)^2}{\delta^2}\right)\omega(f,\delta). \tag{1.6}$$

(iii) *The following equivalence holds: $f \in Lip_M\alpha$ if $\omega(f,\delta) \le M.\delta^\alpha$, where $0 < \alpha \le 1$ and $M > 0$.*

However we also give estimates, where moduli of higher order are involved. Therefore we give the definition of $\omega_k,\ k \in \mathbb{N}$ as given in 1981 by L. L. Schumaker in his book [96]:

Definition 1.20. *For $k \in \mathbb{N}$, $\delta \in \mathbb{R}_+$ and $f \in C[a,b]$ the modulus of smoothness of order k is defined by*

$$\omega_k(f,\delta) := \sup\{|\Delta_h^k f(x)| \ : 0 \le h \le \delta,\ x, x+kh \in [a,b]\}. \tag{1.7}$$

where $\Delta_{h\varphi}^r f$ is the k-th order of forward difference of $f(x)$ with step lenght h.

Remark 1.21. *For clarity sometimes we will write $\omega_k(f,\delta;[a,b])$. It is obvious that for $\delta \ge \frac{b-a}{k}$ one has $\omega_k(f,\delta) = \omega_k(f,\frac{b-a}{k})$.*

We collect in the following proposition some useful properties of ω_k:

Proposition 1.22. *(see [102])*

(i) $\omega_k(f,0) = 0$.

(ii) $\omega_k(f,\cdot)$ *is a positive, continuous and non-decreasing function on* R_+.

(iii) $\omega_k(f,\cdot)$ *is sub-additive, i.e.,* $\omega_1(f,\delta_1+\delta_2) \le \omega_1(f,\delta_1)+\omega_1(f,\delta_2),\ \delta_1,\delta_2 \ge 0.$

(iv) $\forall\delta \ge 0,\ \omega_{k+1} \le 2\omega_k(f,\delta)$.

(v) If $f \in C^1[a,b]$ *then* $\omega_{k+1}(f,\delta) \le \delta\cdot\omega_k(f',\delta),\ \delta \ge 0.$

(vi) If $f \in C^r[a,b]$ *then* $\omega_r(f,\delta) \le \delta^r \sup\limits_{\delta\in[a,b]} |f^{(r)}(\delta)|.$

(vii) $\forall\delta > 0$ *and* $n \in \mathbb{N},\ \omega_k(f,n\delta) \le n^k\omega_k(f,\delta)$.

(viii) $\forall\delta > 0$ *and* $r > 0, \omega_k(f,r\delta) \le (1+[r])^k\omega_k(f,\delta)$, *where* $[a]$ *is the integer part of a.*

(ix) If $\delta \ge 0$ *is fixed, then* $\omega_k(f,\cdot)$ *is a seminorm on* $C[a,b]$.

Corollary 1.23. *(see [102])*

(i) $\forall\delta > 0,\ \omega_{k+r}(f,\delta) \le 2^r\omega_k(f,\delta),\ k,r \in \mathbb{N}.$

(ii) $\forall 0 < \delta \le 1,\ \omega_{k+1}(f,\delta^k) \le \omega_k(f,\delta).$

1.3.2 K-functionals and their relationship to the moduli

J. Peetre [84], in 1968, introduced a functional, which is called *Peetres K-functional*, for investigation of interpolation spaces between two Banach spaces. It represents another important mean to measure the smoothness of a function in terms of how well it can be approximated by smoother functions. It is possible to define the K-functional in a very general context as is presented in [26]. This can be used in applications and in particular for polynomials of best approximation.

The classical definition of the K-functional is given below.

Definition 1.24. *For any* $f \in C[a,b],\ \delta \ge 0$ *and integers* $s \ge 1$ *we call*

$$\begin{aligned} K_s(f,\delta)_{[a,b]} &:= K(f,\delta;C[a,b],C^s[a,b]) \\ &:= \inf\{\|f-g\|_\infty + \delta\cdot\|g^{(s)}\|_\infty : g \in C^s[a,b]\}, \end{aligned} \tag{1.8}$$

as the Peetre's K-functional of order s.

Whenever there is no doubt about the interval of definition of f we shall use for $K_s(f,\delta)_{[a,b]}$ the abbreviation $K_s(f,\delta)$.

It is clear that the quantity in (1.8) reflects some approximation properties of f : the inequality $K_s(f,\delta) < \varepsilon,\ \delta > 0$ implies that f can be approximated with error $\|f-g\|_\infty < \varepsilon$ in $C[a,b]$ by an element $g \in C^s[a,b]$, whose norm is not too large, $\|g^{(s)}\|_\infty \leq \frac{\varepsilon}{\delta}$.

The following lemma collects some of the properties of $K_s(f,\cdot)$. They were proven by Butzer & Berens [15], but they can also be found in more recent work on approximation theory as in: [24], [36] and [96].

Lemma 1.25. *(see Proposition 3.2.3 in [15]) Let $K_s(f,\cdot)$ be defined as in (1.8).*

(i) The mapping $K_s(f,\delta) : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous especially at $\delta = 0$, i.e.,

$$\lim_{\delta\to 0^+} K_s(f,\delta) = 0 = K_s(f,0).$$

(ii) For each $f \in C[a,b]$ the mapping $K_s(f,\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is monotonically increasing and concave function.

(iii) For arbitary $\lambda, \delta \geq 0$, and fixed $f \in C[a,b]$, one has the inequality.

$$K_s(f,\lambda\cdot\delta) \leq \max\{1,\lambda\}\cdot K_s(f,\delta).$$

(iv) For arbitrary $f_1,\ f_2 \in C[a,b]$ we have $K_s(f_1+f_2,\delta) \leq K_s(f_1,\delta)K_s(f_2,\delta),\ \delta \geq 0$.

(v) For each $\delta \geq 0$ fixed, $K_s(\cdot,\delta)$ is a seminorm on $C[a,b]$, such that

$$K_s(f,\delta) \leq \|f\|_\infty,$$

for all $f \in C[a,b]$.

The following theorem establishes the close relationship between the K-functional and the moduli of smoothness. K_s and ω_s are related by the following *equivalence relation*, see H. Johnen [53]:

Theorem 1.26. *There exist constants C_1 and C_2, depending only on s and $[a,b]$ such that*

$$C_1\cdot\omega_s(f,\delta) \leq K_s(f,\delta^s) \leq C_2\cdot\omega_s(f,\delta), \tag{1.9}$$

for all $f \in C[a,b]$ and $\delta > 0$.

In general there are no sharp constants known in the above (double) inequality. However, there are two exceptional cases for $s = 1, 2$ (see [83]).

For the applications we have in mind in this general context, it suffices to consider the case $r = 2$.

For more details, one can refer to [103].

1.3.3 Žuk's function and its applications

Some of the estimates in terms of different moduli of smoothness can be elegantly proven by using as an intermediate a special smoothing function that was constructed by V. Žuk in [111]. Therefore we find it instructive to present here its definition and its relevant properties, see also [37].

Žuk's approach was the following: For $f \in C[a,b]$ he first defined the extension $f_h : [a-h, b+h] \to \mathbb{R}$, with $h > 0$, by

$$f_h(x) := \begin{cases} P_-(x), & a-h \le x \le a, \\ f(x), & a \le x \le b, \\ P_+(x), & b < x \le b+h, \end{cases}$$

where $P_-, P_+ \in \Pi_1$ are *the best approximants* to f on the indicated intervals where $(\Pi_n[a,b],\ n \in \mathbb{N}_0)$ denotes the linear space of all real polynomials with the degree at most n.

Then Žuk defined its function $Z_h f(\cdot)$ (sometimes also denoted by $f_{2,h}(\cdot)$) using the *second order Steklov means*

$$Z_h f(x) := \frac{1}{h}\int_{-h}^{h}\left(1 - \frac{|t|}{h}\right) f_h(x+t)dt, \quad x \in [a,b].$$

It can be shown that $Z_h f \in W_{2,\infty}[a,b]$, where $W_{2,\infty}[a,b]$ denotes the set of all real-valued and continuous functions that verify f' absolutely continuous, $\|f''\|_{L_\infty} < \infty$ and $\|f''\|_{L_\infty} = ess \sup\limits_{x\in[0,1]} |f''|$.

The following estimates were proven in [37] (or [111]).

Lemma 1.27. *Let $f \in C[a,b]$, $0 < h \le \frac{1}{2}(b-a)$. Then*

$$\|f - Z_h f\|_\infty \le \frac{3}{4} \cdot \omega_2(f,h),$$
$$\|(Z_h f)''\|_{L_\infty} \le \frac{3}{2} \cdot h^{-2} \cdot \omega_2(f,h).$$

1.3.4 Korovkin-type approximation

Now we state the *Bohman-Korovkin theorem*, the classical result of theory of approximation. This result provides a necessary and sufficient condition for the convergence of a sequence of positive linear operators towards the identity operator.

Theorem 1.28. *Let $L_n : C[a,b] \to C[a,b]$ be a sequence of positive linear operators. If $\lim_{n\to\infty} L_n e_i = e_i$, $i = 0,1,2$, uniformly on $[a,b]$, then $\lim_{n\to\infty} L_n f = f$ uniformly on $[a,b]$ for every $f \in C[a,b]$.*

Remark 1.29. *Due to the above result the monomials e_j, $j = 0,1,2$, play an important role in the approximation theory by positive linear operators on spaces on continuous functions. They are called Korovkin test-functions.*

This elegant and simple result has inspired many mathematicians to extend the last theorem in different directions, generalizing the notion of sequence and considering different spaces. In this way a special branch of approximation theory arose, called Korovkin-type approximation theory. A complete and comprehensive exposure on this topic can be found in [2].

The function

$$T_n(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \tag{1.10}$$

is called a trigonometric polynomial of order n if $a_n^2 + b_n^2 \neq 0$, and the series

$$\frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots + (a_n \cos nx + b_n \sin nx) + \cdots$$

is called a trigonometric series. Obviously, the product of two trigonometric polynomials of degree m and n, respectively, is a polynomial of degree $m+n$.

Another important result given by Korovkin [61] for periodic functions is the following:

Theorem 1.30. *If the three conditions*

$$L_n(1;x) = 1 + \alpha_n(x),$$

$$L_n(\cos t; x) = \cos x + \beta_n(x),$$

$$L_n(\sin t; x) = \sin x + \gamma_n(x)$$

are satisfied for the sequence of positive linear operators $L_n(f;x)$, where $\alpha_n(x)$, $\beta_n(x)$ and $\gamma_n(x)$ converge uniformly to zero in the interval $[a,b]$, then the sequence $L_n(f;x)$ converges uniformly to the function $f(x)$ in this interval in case the function $f(t)$ is bounded, has period 2π, is continuous on the interval $[a,b]$, continuous on the right at the point b and on the left at the point a.

1.3.5 Local and Global approximation

In [25], it was shown that for the Bernstein operators given by (1.1), the estimate

$$|f(x) - B_n(f;x)| \leq C\omega^2_{\varphi^\lambda}(f, n^{-1/2}\varphi(x)^{1-\lambda}), \quad x \in [0,1] \tag{1.11}$$

holds true, where $\lambda \in [0,1]$, $\varphi(x) = \sqrt{x(1-x)}$, and the Ditzian-Totik modulus of smoothness of second order is given by

$$\omega^2_\phi(f,\delta) := \sup_{|h|\leq\delta} \sup_{x\pm\phi(x)\in[0,1]} |f(x-\phi(x)h) - 2f(x) + f(x+\phi(x)h)|, \tag{1.12}$$

in which $\phi : [0,1] \to \mathbb{R}$ is an admissible step-weight function (for details see [26]).

The case $\lambda = 0$ in (1.11) gives the classical local estimate whereas $\lambda = 1$ gives the global estimate developed by Ditzian and Totik. Therefore (1.11) bridges the gap between local and global approximation theorems for the Bernstein operators (1.1).

For more details about approximation of functions by positive linear operators, one can refer to [39, 65, 97].

1.4 Hyers-Ulam stability of operators

Hyers-Ulam stability is one of the main topics in the theory of functional equations and is connected with perturbation theory and the notion of shadowing in dynamical systems [82]. Recall that an equation is called stable in the Hyers-Ulam sense if for any solution of the perturbed equation, called an approximate solution, there exists a solution of the equation close to it.

The starting point of the stability theory of functional equations was a problem formulated in the celebrated book by Pólya and Szegö [88], and a talk given by S.M. Ulam [107] to a conference at Wisconsin University, Madison in 1940: "Given a metric group $(G, \cdot, \rho)$, a number $\varepsilon > 0$ and a mapping $f : G \to G$ which satisfies the inequality $\rho(f(xy), f(x)f(y)) < \varepsilon$ for all $x, y \in G$, does there exist a homomorphism a of G and a constant $k > 0$, depending only on G, such that $\rho(a(x), f(x)) \leq k\varepsilon$ for all $x \in G$?" If the answer is affirmative the equation $f(xy) = f(x)f(y)$ of the homomorphism is called stable; see [14, 48]. The first answer to Ulam's problem was given by D.H. Hyers [47] in 1941 for the Cauchy functional equation in Banach spaces, more precisely he proved: "Let X, Y be Banach spaces, ε a non-negative number, $f : X \to Y$ a function satisfying $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X$, then there exists a unique additive mapping $a : X \to Y$ with the property $\|f(x) - a(x)\| \leq \varepsilon$ for all $x \in X$." Due to the question of Ulam and the result of Hyers this type of stability is called today Hyers-Ulam stability of functional equations. After Hyers result a large amount of literature was devoted to study the Hyers-Ulam stability for various equations. For definitions, approaches and results on Hyers-Ulam stability we refer the reader to [12, 54, 92, 93].

It seems that the Hyers-Ulam stability of linear operators was considered for the first time in the papers by Miura, Takahasi et al. (see [76]). They obtained a characterization of the Hyers-Ulam stability and a representation of the Hyers-Ulam stability constant for linear operators. For more details about the Hyers-Ulam stability of operators, one may refer to [42, 44, 48].

1.5 Basics of quantum calculus

At the last quarter of 20th century, q-calculus appeared as a coection between mathematics and physics. It has a lot of applications in different mathematical areas, such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions and other sciences-quantum theory, mechanics and theory of relativity.

Consider the following expression:

$$\frac{f(x) - f(x_0)}{x - x_0}.$$

As x approaches x_0, the limit, if it exists, gives the familiar definition of the derivative $\frac{df}{dx}$ of a function $f(x)$ at $x = x_0$. However, if we take $x = qx_0$, where q ia a fixed number different from 1, and do not take the limit, we enter the fascinating world of quantum calculus. In the course of developing quantum calculus along the traditional lines of ordinary calculus it is discovered many important notions and results in the above discussed fields of mathematics.

For example, the q-derivative of x^n is $[n]_q x^{n-1}$, where

$$[n]_q = \frac{1-q^n}{1-q}$$

is the q-analogue of integer n (in the sense that n is the limit of $[n]_q$ as $q \to 1$).

We have also a generalization of q-calculus with one more parameter, we can say it is a two parameter quantum calculus. Generally it is called (p, q)-calculus.

1.5.1 Basics of q-calculus

The applications of q-calculus emerged as a new area in the field of approximation theory from last two decades. The development of q-calculus has led to the discovery of various modifications of Bernstein polynomials involving q-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations.

The q-integer $[n]_q$, the q-factorial $[n]_q!$ and the q-binomial coefficient are defined by (see [7, 55])

$$[n]_q := \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \in \mathbb{R}^+ \setminus \{1\} \\ n, & \text{if } q = 1, \end{cases} \quad \text{for } n \in \mathbb{N} \text{ and } [0]_q = 0,$$

$$[n]_q! := \begin{cases} [n]_q[n-1]_q \cdots [1]_q, & n \geq 1, \\ 1, & n = 0, \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!},$$

respectively.

The q-binomial coefficient satisfies the following recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \quad \text{and}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q.$$

The q-analogue of $(1+x)^n$ is the polynomial

$$(1+x)_q^n := \begin{cases} (1+x)(1+qx)\cdots(1+q^{n-1}x) & n = 1, 2, 3, \cdots \\ 1 & n = 0. \end{cases}$$

A q-analogue of the common Pochhammer symbol also called a q-shifted factorial is defined as

$$(x;q)_0 = 1,\ (x;q)_n = \prod_{j=0}^{n-1}(1-q^j x),\ (x;q)_\infty = \prod_{j=0}^{\infty}(1-q^j x).$$

The Gauss binomial formula:

$$(x+a)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} a^k x^{n-k}. \tag{1.13}$$

The Heine's binomial formula:

$$\frac{1}{(1-x)_q^n} = 1 + \sum_{k=1}^{\infty} \frac{[n]_q [n+1]_q \cdots [n+k-1]_q}{[k]_q!} x^k. \tag{1.14}$$

The q-derivative $D_q f$ of a function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x},\ \text{ if } x \neq 0, \tag{1.15}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

Note that $\lim\limits_{q \to 1} D_q f(x) = \frac{df(x)}{dx}$, if f is differentiable. Clearly, for $n \geq 1$, we can show that

$$D_q(1+x)_q^n = [n]_q(1+qx)_q^{n-1} \text{ and } D_q\left\{\frac{1}{(1+x)_q^n}\right\} = -\frac{[n]_q}{(1+x)_q^{n+1}}. \tag{1.16}$$

We can notice that

(i) The q-derivative of a function is a linear operator.

(ii) The q-derivative of a product at $x \neq 0$ is

$$\begin{aligned} D_q\big(f(x)g(x)\big) &= f(x)D_q g(x) + D_q f(x) g(qx), \\ &= f(qx) D_q g(x) + D_q f(x) g(x). \end{aligned}$$

(iii) The Leibniz rule for the q-derivative operator is defined as

$$D_q^{(n)}(fg)(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q D_q^{(k)} f(q^{n-k}x) D_q^{(n-k)} g(x). \tag{1.17}$$

(iv)

$$D_q\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(qx)D_qf(x) - f(qx)D_qg(x)}{g(x)g(qx)}. \tag{1.18}$$

A q-analogue of classical exponential function e^x is

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} = \frac{1}{\left(1-(1-q)x\right)_q^{\infty}}. \tag{1.19}$$

Another q-analogue of classical exponential function is

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]_q!} = \left(1+(1-q)x\right)_q^{\infty}. \tag{1.20}$$

The exponential functions satisfy the following relation

(i) $D_qe_q(x) = e_q(x), \quad D_qE_q(x) = E_q(qx).$

(ii) $e_q(x)E_q(-x) = E_q(x)e_q(-x) = 1.$

Note that for $q \in (0,1)$, the series expansion of $e_q(x)$ has radius of convergence $\frac{1}{1-q}$. On the contrary, the series expansion of $E_q(x)$ converges for every real x.

The Jackson definite integral ([51, 106]) of the function f is defined by

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \quad a \in \mathbb{R}. \tag{1.21}$$

Notice that the series on the right-hand side is guaranteed to be convergent as soon as the function f is such that for some $C > 0,\ \alpha > -1,\ |f(x)| < Cx^{\alpha}$ in a right neighborhood of $x = 0$.

The q-improper integral is defined as (see Koornwinder [59])

$$\int_0^{\infty/A} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A}, \quad A > 0. \tag{1.22}$$

If the function f satisfies the conditions $|f(x)| < Cx^{\alpha},\ \forall x \in [0,\varepsilon)$, for some $C > 0,\ \alpha > -1,\ \varepsilon > 0$ and $|f(x)| < Dx^{\beta},\ \forall x \in [N,\infty)$, for some $D > 0,\ \beta < -1,\ N > 0$, then the series on the right hand side is convergent. In general even though when these conditions are satisfied, the value of sum on the right hand side of (1.22) will be dependent on the constant A. In order to get the integral independent of A, in the anti q-derivative, we have to take the limits as $x \to 0$ and $x \to 1$, respectively.

The q-gamma function defined by

$$\Gamma_q(t) = \int_0^{1/1-q} x^{t-1} E_q(-qx) d_q x, \quad t > 0$$

satisfies the following functional equation:

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t), \tag{1.23}$$

where $[t]_q = \frac{1-q^t}{1-q}$ and $\Gamma_q(1) = 1$.

The q-Beta function is defined as follows

$$B_q(t,s) = K(A,t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}} d_q x,$$

where

$$K(x,t) = \frac{1}{x+1} x^t \left(1 + \frac{1}{x}\right)_q^t (1+x)_q^{1-t},$$

and also

$$K(A, t+1) = q^t K(A,t), \ A > 0. \tag{1.24}$$

In particular for any positive integer n,

$$K(A,n) = q^{\frac{n(n-1)}{2}}, \quad K(A,0) = 1 \quad \text{and}$$

$$B_q(t;s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}. \tag{1.25}$$

One can find more results on integral representations of q-gamma and q-beta functions in [23].

1.5.2 Basics of (p,q)-calculus

The (p,q)-integers were introduced in order to generalize or unify several forms of q-oscillator algebras well known in the earlier physics literature related to the representation theory of single parameter quantum algebras, i.e., q-calculus.

Let us recall certain notions of (p,q)-calculus.

The (p,q)-integer $[n]_{p,q}$, and the (p,q)-factorial are defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \cdots, \quad 0 < q < p \leq 1 \quad \text{and}$$

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}!, \quad n \geq 1, \quad [0]_{p,q}! = 1.$$

The (p,q)-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad 0 \leq k \leq n.$$

The (p,q)-analogue of binomial expansion $(ax + by)^n$ is defined as

$$(ax + by)^n_{p,q} := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}} a^{n-k} b^k x^{n-k} y^k, \tag{1.26}$$

$$(x - y)^n_{p,q} = (x - y)(px - qy)(p^2x - q^2y) \cdots (p^{n-1}x - q^{n-1}y). \tag{1.27}$$

Let m and n be two non-negative integers. Then the following assertion is valid

$$(x - y)^{m+n}_{p,q} := (x - y)^m_{p,q} (p^m x - q^m y)^n_{p,q}. \tag{1.28}$$

The (p,q)-derivative of a function f, denoted by $D_{p,q}f$, is defined by

$$(D_{p,q}f)(x) := \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0, \quad (D_{p,q}f)(0) := f'(0) \tag{1.29}$$

provided that f is differentiable at 0. It can be easily seen that $D_{p,q}x^n = [n]_{p,q}x^{n-1}$. The (p,q)-derivative fulfils the following product rules

$$\begin{aligned} D_{p,q}(f(x)g(x)) &:= f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x), \\ D_{p,q}(f(x)g(x)) &:= f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x). \end{aligned}$$

Moreover,

$$\begin{aligned} D_{p,q}\left(\frac{f(x)}{g(x)}\right) &:= \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}, \\ D_{p,q}\left(\frac{f(x)}{g(x)}\right) &:= \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}. \end{aligned}$$

We consider the (p, q)-exponential functions in the following forms:

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{n(n-1)/2} \frac{x^n}{[n]_{p,q}!}, \tag{1.30}$$

$$E_{p,q}(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_{p,q}!}, \tag{1.31}$$

which satisfy the equality $e_{p,q}(x)E_{p,q}(-x) = 1$.

The (p, q)-definite integrals of the function f are defined by

$$\int_0^a f(x)d_{p,q}x = (q-p)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}a\right), \quad \text{when} \quad \left|\frac{p}{q}\right| < 1, \tag{1.32}$$

and

$$\int_0^a f(x)d_{p,q}x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}a\right), \quad \text{when} \quad \left|\frac{p}{q}\right| > 1. \tag{1.33}$$

It is very clear that q-integers and (p, q)-integers are different. For instance, we cannot obtain (p, q)-integers just by replacing q by $\frac{q}{p}$ in the definition of q-integers. For $p = 1$, all the notions of (p, q)-calculus are reduced to q-calculus. So there is no confusion to say that (p, q)-calculus is a consequence of q-calculus. For details on (p, q)-calculus, one can refer to [46, 52, 57, 94, 95].

Chapter 2

Hyers-Ulam stability of some positive linear operators

2.1 Introduction

The Hyers-Ulam stability of linear operators was considered for the first time in the papers by Miura, Takahasi et al. (see [42, 44, 76]). Similar type of results are obtained in [105] for weighted composition operators on $C(X)$, where X is a compact Hausdorff space. A result on the stability of a linear composition operator of the second order was given by J. Brzdek and S.M. Jung in [13].

Recently, Popa and Raşa [89, 90] have obtained a result on the Hyers-Ulam stability of some operators from approximation theory and they have shown that Bernstein operators, Stancu operators and Kantorovich operators are Hyers-Ulam stable and they also found the best HU-constants for the respective operators. They also proved some of the operators e.g., Szász-Mirakjan operators, Beta operators, are not stable in the sense of Hyers and Ulam.

The present chapter has been divided in four sections. The first two sections are about introduction and Hyers-Ulam stability property of the operators. In the third section we have shown that Bernstein-Stancu type operators, Kantorovich-Stancu type operators, Kantorovich-Bernstein-Stancu type operators with shifted knots and an operator introduced by J.P. King are Hyers-Ulam stable. Further we find the best Hyers-Ulam stability constants for some of these operators. We also prove that Szász-Mirakjan and Kantorovich-Szász-Mirakjan type operators are unstable in the sense of Hyers and Ulam. The last section of this chapter is devoted to show the Hyers-Ulam stability of Bernstein-Schurer operators and Kantorovich-Schurer operators on a compact disk. Further, we also show that complex Lorentz polynomials are not stable in the sense of Hyers and Ulam.

Let $C[0,1]$ be the space of all continuous, real-valued functions defined on $[0,1]$, and

$C_B[0, +\infty)$ the space of all continuous, bounded, real-valued functions on $[0, +\infty)$. Endowed with the supremum norm, they are Banach spaces.

2.2 The Hyers-Ulam stability property of operators

In this section, we recall some basic definitions and results on Hyers-Ulam stability property which form the background of our main results.

Definition 2.1. *Let A and B be normed spaces and T a mapping from A into B. We say that T has the Hyers-Ulam stability property (briefly, T is HU-stable) [105] if there exists a constant K such that:*
(i) for any $g \in T(A)$, $\varepsilon > 0$ and $f \in A$ with $\|Tf - g\| \leq \varepsilon$, there exists an $f_0 \in A$ such that $Tf_0 = g$ and $\|f - f_0\| \leq K\varepsilon$.

The number K is called a *HUS constant of T*, and the infimum of all HUS constants of T is denoted by K_T. Generally, K_T is not a HUS constant of T (see [42, 44]).

Let now T be a bounded linear operator with the kernel denoted by $N(T)$ and the range denoted by $R(T)$. Consider the one-to-one operator $\widetilde{T}$ from the quotient space $A/N(T)$ into B:

$$\widetilde{T}(f + N(T)) = Tf, \;\; f \in A,$$

and the inverse operator $\widetilde{T}^{-1} : R(T) \to A/N(T)$ be the inverse of $\widetilde{T}$.

In [105] the authors established the following result:

Theorem 2.2. *([105]) Let A and B be Banach spaces and $T : A \to B$ be a bounded linear operator. Then the following statements are equivalent:*
(a) T is HU-stable;
(b) $R(T)$ is closed;
(c) $\widetilde{T}^{-1}$ is bounded.
Moreover, if one of the conditions (a), (b), (c) is satisfied, then

$$K_T = \|\widetilde{T}^{-1}\|.$$

Remark 2.3. *([89]) (1) Condition (i) of Def. 2.1 expresses the Hyers-Ulam stability of the equation $Tf = g$, where $g \in R(T)$ is given and $f \in A$ is unknown.*
(2) If $T : A \to B$ is a bounded linear operator, then (i) is equivalent to:
(ii) for any $f \in A$ with $\|Tf\| \leq 1$ there exists an $f_0 \in N(T)$ such that

$$\|f - f_0\| \leq K, \;\; \text{(see [44])}.$$

So, in what follows, we shall study the HU-stability of a bounded linear operator $T : A \to B$ by checking the existence of a constant K for which (ii) is satisfied, or equivalently, by checking the boundedness of $\widetilde{T}^{-1}$.

The main results used in our approach for obtaining, in some concrete cases, the explicit value of K_T are the formula given above and a result by Lubinsky and Ziegler [66] concerning coefficient bounds in the Lorentz representation of a polynomial.

Let $p \in \Pi_n$, where Π_n is the set of all polynomials of degree at most n with real coefficients. Then p has a unique Lorentz representation of the form

$$p(x) = \sum_{k=0}^{n} c_k x^k (1-x)^{n-k}, \tag{2.1}$$

where $c_k \in \mathbb{R}$, $k = 0, 1, \cdots, n$. While it is not unique in general - for example

$$1 = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{any } n \geqq 0,$$

– it becomes unique if we insist in (2.1) that n equals the degree of p.

Remark that, in fact, it is a representation in Bernstein-Bézier basis. Let T_n denote the usual nth degree Chebyshev polynomial of the first kind. Then the following representation holds (see [66]):

$$T_n(2x-1) = \sum_{k=0}^{n} d_{n,k} x^k (1-x)^{n-k} (-1)^{n-k}, \tag{2.2}$$

where

$$d_{n,k} := \sum_{j=0}^{\min\{k,n-k\}} \binom{n}{2j} \binom{n-2j}{k-j} 4^j, \quad k = 0, 1, \cdots, n. \tag{2.3}$$

It is proved in [89] that $d_{n,k} = \binom{2n}{2k}$, $k = 0, 1, \cdots, n$. Therefore

$$T_n(2x-1) = \sum_{k=0}^{n} \binom{2n}{2k} (-1)^{n-k} x^k (1-x)^{n-k}.$$

Theorem 2.4. *(Lubinsky and Ziegler [66]) Let $p(x)$ have the representation (2.1), and let $0 \le k \le n$. Let $d_{n,k}$ be defined by (2.3). Then*

$$|c_k| \le d_{n,k} \cdot \|p\|_{L_\infty[0,1]}$$

with equality if and only if $p(x)$ is a constant multiple of $T_n(2x-1)$.

As in [66], we observe that

$$\|p\|_{L_\infty[0,1]} \leqq \max_{0\leqq k\leqq n}\left\{|c_k|/\binom{n}{k}\right\}, \tag{2.4}$$

where

$$L_\infty[0,1] = \{f : [0,1] \to \mathbb{R} : \operatorname{ess\,sup}|f| < \infty\},$$

the essential supremum of f on $[0,1]$ is defined by

$$\operatorname{ess\,sup}|f(x)| = \inf\{K : |f(x)| \leq K \text{ a.e. on } [0,1]\}.$$

Let A be the Banach space and M the closed subspace of A, then by A/M, we denote the quotient space with the usual norm;

$$\|f + M\| = \inf_{h\in M}\|f + h\|. \tag{2.5}$$

For more details, one can refer to [42].

2.3 HU-stability of operators and their HU-constants

(i) *Kantorovich-Stancu Type Operators*

For each integer $n \geq 1$, let Π_n be the subspace of $C[0,1]$ consisting of all polynomial functions of degree $\leq n$. Let $C[0,1+m]$ be the linear space of all continuous functions $f : [0,1] \to \mathbb{R}$, endowed with the supremum norm denoted by $\|.\|$, and a, b real numbers, $0 \leq a \leq b$. The Kantorovich-Stancu type operator ([8]) $K_{n,m} : C[0,1+m] \to \Pi_{n+m}$ is defined by

$$K_{n,m}(f)(x) \quad = (n+m+\beta+1)\sum_{k=0}^{n+m}\binom{n+m}{k}x^k(1-x)^{n+m-k}\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(t)dt,$$

$f \in C[0,1+m]$. The kernal of $K_{n,m}$ is given by

$$N(K_{n,m}) = \left\{f \in C[0,1+m] : \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(t)dt = 0,\ \ 0 \leq k \leq n+m\right\}.$$

which is a closed subspace of $C[0,1+m]$, and $R(K_{n,m}) = \Pi_{n+m}$. The operator $\widetilde{K}_{n,m}$: $C[0,1+m]/N(K_{n,m}) \to \Pi_{n+m}$ is bijective, $\widetilde{K}_{n,m}^{-1} : \Pi_{n+m} \to C[0,1+m]/N(K_{n,m})$ is bounded since $\dim \Pi_{n+m} = n+m+1$, so according to Theorem 2.2 the operator $K_{n,m}$ is Hyers-Ulam stable.

Theorem 2.5. *For* $n \geq 1$

$$K_{K_{n,m}} = \|\widetilde{K}_{n,m}^{-1}\| = \frac{(n+\beta+1)\binom{2(n+m)}{2[\frac{n+m}{2}]}}{(n+m+\beta+1)\binom{n+m}{[\frac{n+m}{2}]}}.$$

Proof. Let $p \in \Pi_{n+m}$, $\|p\| \leq 1$, and its Lorentz representation

$$p(x) = \sum_{k=0}^{n+m} c_k(p) x^k (1-x)^{n+m-k}, \quad x \in [0,1].$$

Consider the piecewise constant function $f_p \in C[0, 1+m]$ defined by

$$f_p(t) = \begin{cases} \frac{(n+\beta+1)c_k(p)}{(n+m+\beta+1)\binom{n+m}{k}}, & t \in [\frac{k+\alpha}{n+\beta+1}, \frac{k+\alpha+1}{n+\beta+1}); \\ c_{n+m}(p), & t \in [\frac{n+m}{n+\beta+1}, 1]; \end{cases}$$

$0 \leq k \leq n+m-1$. Then $K_{n,m} f_p = p$ and $\widetilde{K}_{n,m}^{-1}(p) = f_p + N(K_{n,m})$.

As usual, the norm of $\widetilde{K}_{n,m}^{-1} : \Pi_{n+m} \to C[0,1+m]/N(K_{n,m})$ is defined by

$$\|\widetilde{K}_{n,m}^{-1}\| = \sup_{\|p\| \leq 1} \|\widetilde{K}_{n,m}^{-1}(p)\| = \sup_{\|p\| \leq 1} \inf_{h \in N(K_{n,m})} \|f_p + h\|.$$

Clearly

$$\inf_{h \in N(K_{n,m})} \|f_p + h\| = \|f_p\| = \max_{0 \leq k \leq n+m} \frac{(n+\beta+1)|c_k(p)|}{(n+m+\beta+1)\binom{n+m}{k}}.$$

Therefore

$$\begin{aligned} \|\widetilde{K}_{n,m}^{-1}\| &= \sup_{\|p\| \leq 1} \max_{0 \leq k \leq n+m} \frac{(n+\beta+1)|c_k(p)|}{(n+m+\beta+1)\binom{n+m}{k}} \\ &\leq \sup_{\|p\| \leq 1} \max_{0 \leq k \leq n+m} \frac{(n+\beta+1)\|p\| d_{n+m,k}}{(n+m+\beta+1)\binom{n+m}{k}} \\ &= \max_{0 \leq k \leq n+m} \frac{(n+\beta+1) d_{n+m,k}}{(n+m+\beta+1)\binom{n+m}{k}}. \end{aligned}$$

On the other hand, let $q(x) = T_n(2x-1)$, $x \in [0,1]$. Then $\|q\| = 1$ and $|c_k(q)| = d_{n+m,k}$, $0 \leq k \leq n+m$, according to Theorem 2.4. Consequently

$$\|\widetilde{K}_{n,m}^{-1}\| \geq \max_{0 \leq k \leq n+m} \frac{(n+\beta+1)|c_k(q)|}{(n+m+\beta+1)\binom{n+m}{k}} = \max_{0 \leq k \leq n+m} \frac{(n+\beta+1) d_{n+m,k}}{(n+m+\beta+1)\binom{n+m}{k}}$$

and so

$$\|\widetilde{K}_{n,m}^{-1}\| = \max_{0 \leq k \leq n+m} \frac{(n+\beta+1) d_{n+m,k}}{(n+m+\beta+1)\binom{n+m}{k}} = \max_{0 \leq k \leq n+m} \frac{(n+\beta+1)\binom{2(n+m)}{2k}}{(n+m+\beta+1)\binom{n+m}{k}}.$$

Let

$$a_k = \frac{(n+\beta+1)\binom{2(n+m)}{2k}}{(n+m+\beta+1)\binom{n+m}{k}}, \; 0 \le k \le n+m.$$

Then

$$\frac{a_{k+1}}{a_k} = \frac{2n+2m-2k-1}{2k+1}, \;\; 0 \le k \le n+m.$$

The inequality $\frac{a_{k+1}}{a_k} \ge 1$ is satisfied if and only if $k \le [\frac{n+m-1}{2}]$, therefore

$$\max_{0 \le k \le n+m} a_k = a_{[\frac{n+m-1}{2}]+1} = \begin{cases} a_{[\frac{n+m}{2}]}, & n+m \text{ is even;} \\ a_{[\frac{n+m}{2}]+1}, & n+m \text{ is odd.} \end{cases}$$

Since $a_{[\frac{n+m}{2}]+1} = a_{[\frac{n+m}{2}]}$ if $n+m$ is an odd number, we conclude that

$$K_{K_{n,m}} = \|\widetilde{K}_{n,m}^{-1}\| = \frac{(n+\beta+1)\binom{2(n+m)}{2[\frac{n+m}{2}]}}{(n+m+\beta+1)\binom{n+m}{[\frac{n+m}{2}]}}.$$

This completes the proof of the theorem. □

Remark 2.6. *For $m = \beta = 0$, the above operator reduces to the classical Kantorovich operator. Therefore the infimum of the HUS constant of the Kantorovich operator is*

$$K_{K_n} = \binom{2n}{2[\frac{n}{2}]} / \binom{n}{[\frac{n}{2}]}.$$

This corrects the value provided by Theorem 3.3 in ([90]).

(ii) *King's operator*

Let $\{r_n(x)\}$ be a sequence of continuous functions defined on $[0,1]$ with $0 \le r_n(x) \le 1$, i.e., $r_n(x) : [0,1] \to [0,1]$ are continuous functions. In ([58]) King defined the following interesting sequence of positive linear operators $V_n : C[0,1] \to C[0,1]$ which generalize the classical Bernstein operator B_n defined by

$$V_n f(x) = \sum_{k=0}^{n} \binom{n}{k} (r_n(x))^k (1 - r_n(x))^{n-k} f\left(\frac{k}{n}\right)$$

for all $f \in C[0,1]$, $0 \le x \le 1$. The kernal of V_n is given by

$$N(V_n) = \left\{ f \in C[0,1] : \; f\left(\frac{k}{n}\right) = 0, \;\; 0 \le k \le n \right\}.$$

which is a closed subspace of $C[0,1]$, and $R(V_n) = \Pi_n$. The operator $\widetilde{K}_n : C[0,1]/N(V_n) \to \Pi_n$ is bijective, $\widetilde{V}_n^{-1} : \Pi_n \to C[0,1]/N(V_n)$ is bounded since dim $\Pi_n = n+1$, so according to Theorem 2.2 the operator V_n is Hyers-Ulam stable.

Theorem 2.7. *For $n \geq 1$,*

$$K_{V_n} = \|\widetilde{V}_n^{-1}\| = \binom{2(n)}{2[\frac{n}{2}]} / \binom{n}{[\frac{n}{2}]}.$$

Proof. The proof is similar as Theorem 3.3 for finding the best constant in the case of Bernstein operator in ([89]). □

(iii) *Szász-Mirakjan type operators*

In ([68]), Lupaş proposed the positive linear operators

$$L_n f(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right), \quad x \in [0, +\infty)$$

with the help of the identity $\frac{1}{(1-a)^\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} a^k$, $|a| < 1$ where $(\alpha)_0 = 1$, $(\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1)$; $k \geq 1$. Here we are taking an nth Szász-Mirakjan type operator $L_n : C_B[0, +\infty) \to C_B[0, +\infty)$ defined by

$$L_n f(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{e^k k!} f\left(\frac{k}{n}\right), \quad x \in [0, +\infty).$$

Theorem 2.8. *For $n \geq 1$ the operator L_n is not stable in the sense of Hyers-Ulam.*

Proof. Suppose that for a certain $n \geq 1$, the operator L_n is HU-stable. Then there exists a constant K such that for any $f \in C_B[0, +\infty)$ with $\|L_n f\|_\infty \leq 1$ there exists $g \in N(L_n)$ with $\|f - g\| \leq K$.

According to Stirling's formula, $\lim_{k\to\infty} \frac{k^k}{k! e^k} = 0$, so that there exists a $j \geq 1$ such that $(K+1)\frac{j^j}{j! e^j} \leq 1$.

Let $f \in C_B[0, +\infty)$ be the function defined by

$$f(x) = \begin{cases} 0, & x \in [0, \frac{j-1}{n}] \cup [\frac{j+1}{n}, +\infty); \\ K+1, & x = \frac{j}{n}. \end{cases}$$

f is linear on $[\frac{j-1}{n}, \frac{j}{n}]$ and $[\frac{j}{n}, \frac{j+1}{n}]$. Then

$$\begin{aligned} L_n f(x) &= e^{-nx} \frac{(K+1)}{j! e^j} (nx)_j \\ &= e^{-nx} \frac{(K+1)}{j!} \frac{(nx)(nx+1) \cdots (nx+j-1)}{e^j} \\ &= e^{-nx} \frac{(K+1)}{j!} \frac{n^j x^j (1 + \frac{1}{nx})(1 + \frac{2}{nx}) \cdots (1 + \frac{j-1}{nx})}{e^j} \\ &= e^{-nx} (K+1) \frac{n^j}{j! e^j} (1 + \frac{1}{nx})(1 + \frac{2}{nx}) \cdots (1 + \frac{j-1}{nx}) x^j \end{aligned}$$

$x \in [0, +\infty)$.

It is easy to check that $\|L_n f\|_\infty \leq 1$, so that there exists $g \in N(L_n)$ with $\|f - g\|_\infty \leq K$. But then $g(\frac{j}{n}) = 0$ and consequently

$$K \geq \|f - g\|_\infty \geq |f(\frac{j}{n}) - g(\frac{j}{n})| = K + 1$$

a contradiction. Thus the theorem is proved. □

(iv) *Kantorovich-Szász-Mirakjan type operators*

The classical nth Szász-Mirakjan operators $L_n : C_B[0, +\infty) \to C_B[0, +\infty)$ are defined by (see [2], pp. 338)

$$L_n f(x) = e^{-nx} \sum_{i=0}^{\infty} f\left(\frac{i}{n}\right) \frac{n^i}{i!} x^i, \quad x \in [0, +\infty).$$

The Kantorovich version of the Szász-Mirakjan operators are defined by

$$K_n f(x) = n e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k! e^k} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \quad x \in [0, +\infty).$$

Theorem 2.9. *For $n \geq 1$ the operator K_n is unstable in the sense of Hyers and Ulam.*

Proof. The proof is similar as Theorem 3.3. But here the mapping $f \in C_b[0, +\infty)$ is defined by

$$f(x) = \begin{cases} 0, & x \in [0, \frac{j-1}{n}] \cup [\frac{j+1}{n}, +\infty); \\ \frac{K+1}{n}, & x \in [\frac{j}{n}, \frac{j+1}{n}]. \end{cases}$$

Of course f is linear on $[\frac{j-1}{n}, \frac{j}{n}]$ and $[\frac{j}{n}, \frac{j+1}{n}]$. □

(v) *Other operators*

Consider the operators (a) Bernstein-Stancu type with shifted knots introduced in ([31]) is defined by

$$S_{n,\alpha,\beta}(f; x) = \left(\frac{n + \beta_2}{n}\right)^n \sum_{k=0}^{n} f\left(\frac{k + \alpha_1}{n + \beta_1}\right) \binom{n}{k} \left(x - \frac{\alpha_2}{n + \beta_2}\right)^k \left(\frac{n + \alpha_2}{n + \beta_2} - x\right)^{n-k}$$

where $\frac{\alpha_2}{n+\beta_2} \leq x \leq \frac{n+\alpha_2}{n+\beta_2}$ and α_k, β_k $(k = 1, 2)$ are positive real numbers provided $0 \leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2$;

(b) Kantorovich type generalization of Bernstein-Stancu type operator with shifted knots introduced in ([49]) is defined by

$$S^*_{n,\alpha,\beta}(f;x) = (n+\beta_1+1)\left(\frac{n+\beta_2}{n}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(x-\frac{\alpha_2}{n+\beta_2}\right)^k\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-k}\int_{\frac{k+\alpha_1}{n+\beta_1+1}}^{\frac{k+\alpha_1+1}{n+\beta_1+1}} f(t)dt$$

where $\frac{\alpha_2}{n+\beta_2} \leq x \leq \frac{n+\alpha_2}{n+\beta_2}$ and α_k, β_k $(k = 1,2)$ are positive real numbers provided $0 \leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2$, and

(c) r-th order of the Kantorovich type generalization of Bernstein-Stancu type operator introduced in ([49]) is defined by

$$S^*_{n,\alpha,\beta,r}(f;x) = (n+\beta_1+1)\left(\frac{n+\beta_2}{n}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(x-\frac{\alpha_2}{n+\beta_2}\right)^k\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-k}\int_{\frac{k+\alpha_1}{n+\beta_1+1}}^{\frac{k+\alpha_1+1}{n+\beta_1+1}} \sum_{j=0}^{r} f^{(j)}(t)\frac{(x-t)^j}{j!}dt$$

where $f \in C^r[0,1]$ $(r = 0,1,2,\cdots)$ the set of all functions f having continuous r-th derivative $f^{(r)}(f^{(0)}(x) = f(x))$ on the segment $[0,1]$.

They are Hyers-Ulam stable since their ranges are finite dimensional.

2.4 HU-stability of operators on a compact disk

In this section, we show the Hyers-Ulam stability of some other operators on a compact disk. Let D_R denote the compact disk having radius R, i.e., $D_R = \{z \in \mathbb{C} : |z| \leq R\}$.

(i) *Bernstein-Schurer Operators*

Let $X_{D_R} = \{f : D_R \to \mathbb{C}$ be analytic in $D_R\}$ be the collection of all analytic functions endowed with the supremum norm denoted by $\|.\|$. The complex Bernstein-Schurer operator ([3])

$$S_{n,p}(f)(z) = \sum_{k=0}^{n+p}\binom{n+p}{k}z^k(1-z)^{n+p-k}f\left(\frac{k}{n}\right), \quad z \in \mathbb{C}, \ f \in X_{D_R}.$$

We have $N(S_{n,p}) = \{f \in X_{D_R} : \ f(\frac{k}{n}) = 0, \ 0 \leq k \leq n+p\}$, which is a closed subspace of X_{D_R}, and $R(S_{n,p}) = \Pi_{n+p}$. The operator $\widetilde{S}_{n,p} : X_{D_R}/N(S_{n,p}) \to \Pi_{n+p}$ is bijective, $\widetilde{S}^{-1}_{n,p} : \Pi_{n+p} \to X_{D_R}/N(S_{n,p})$ is bounded since dim $\Pi_{n+p} = 2(n+p+1)$. So according to Theorem 2.2 the operator $S_{n,p}$ is Hyers-Ulam stable.

Theorem 2.10. *For $n \geq 1$, the Hyers-Ulam stability best constant (by Def. 2.1 and Theorem 2.2) is given by*

$$K_{S_{n,m}} = \|\widetilde{S}_{n,m}^{-1}\| = \binom{2(n+m)}{2[\frac{n+m}{2}]} \Big/ \binom{n+m}{[\frac{n+m}{2}]}.$$

Proof. Let $p(z) \in \Pi_{n+m}$, $\|p\| \leq 1$, and its Lorentz representation

$$p(z) = \sum_{k=0}^{n+m} c_k(p) z^k (1-z)^{n+m-k}, \quad |z| \leq R.$$

Consider the constant function $f_p \in X_{D_R}$ defined by

$$f_p\left(\frac{k}{n}\right) = \frac{c_k(p)}{\binom{n+m}{k}}, \quad 0 \leq k \leq n+m.$$

Then $S_{n,m} f_p = p$ and $\widetilde{S}_{n,m}^{-1}(p) = f_p + N(S_{n,m})$ (see [90]). Clearly by (2.4), we have

$$\inf_{h \in N(S_{n,m})} \|f_p + h\| = \|f_p\| = \max_{0 \leq k \leq n+m} |c_k(p)| \Big/ \binom{n+m}{k}.$$

Hence using the above equality, we have

$$\begin{aligned}
\|\widetilde{S}_{n,m}^{-1}\| &= \sup_{\|p\| \leq 1} \|\widetilde{S}_{n,m}^{-1}(p)\| = \sup_{\|p\| \leq 1} \inf_{h \in N(S_{n,m})} \|f_p + h\| \\
&= \sup_{\|p\| \leq 1} \max_{0 \leq k \leq n+m} |c_k(p)| \Big/ \binom{n+m}{k} && (2.6) \\
&\leq \sup_{\|p\| \leq 1} \max_{0 \leq k \leq n+m} d_{n+m,k} \cdot \|p\| \Big/ \binom{n+m}{k} \quad \text{(using Theorem (2.4))} \\
&= \max_{0 \leq k \leq n+m} d_{n+m,k} \Big/ \binom{n+m}{k}. && (2.7)
\end{aligned}$$

On the other hand, let $r(z) = T_n(2z-1)$, $|z| \leq R$. Then $\|r\| = 1$ and $|c_k(r)| = d_{n+p,k}$, $0 \leq k \leq n+p$, according to Theorem 2.4. Consequently by (2.6), we have

$$\|\widetilde{S}_{n,m}^{-1}\| \geq \max_{0 \leq k \leq n+m} |c_k(r)| \Big/ \binom{n+m}{k} = \max_{0 \leq k \leq n+m} d_{n+m,k} \Big/ \binom{n+m}{k} \qquad (2.8)$$

and so by (2.7) and (2.8), we obtain

$$\|\widetilde{S}_{n,m}^{-1}\| = \max_{0 \leq k \leq n+m} \frac{d_{n+m,k}}{\binom{n+m}{k}} = \max_{0 \leq k \leq n+m} \frac{\binom{2(n+m)}{2k}}{\binom{n+m}{k}}.$$

Let

$$a_k = \frac{\binom{2(n+m)}{2k}}{\binom{n+m}{k}}, \quad 0 \leq k \leq n+m.$$

Then

$$\frac{a_{k+1}}{a_k} = \frac{2n+2m-2k-1}{2k+1}, \ \ 0 \le k \le n+m.$$

The inequality $\frac{a_{k+1}}{a_k} \ge 1$ is satisfied if and only if $k \le [\frac{n+m-1}{2}]$, therefore

$$\max_{0 \le k \le n+m} a_k = a_{[\frac{n+m-1}{2}]+1} = \begin{cases} a_{[\frac{n+m}{2}]}, & n+m \text{ is even;} \\ a_{[\frac{n+m}{2}]+1}, & n+m \text{ is odd.} \end{cases}$$

Since $a_{[\frac{n+m}{2}]+1} = a_{[\frac{n+m}{2}]}$ if $n+m$ is an odd number, we conclude that

$$K_{S_{n,m}} = \|\widetilde{S}_{n,m}^{-1}\| = \binom{2(n+m)}{2[\frac{n+m}{2}]} \Big/ \binom{n+m}{[\frac{n+m}{2}]}.$$

This completes the proof of the theorem. □

(ii) *Kantrovich-Schurer Operators*

Let $X_{D_R} = \{f : D_R \to \mathbb{C}$ be analytic in $D_R\}$ be the collection of all analytic functions endowed with the supremum norm denoted by $\|.\|$. The complex Kantrovich-Schurer operator ([3]) $K_{n,m} : X_{D_R} \to \Pi_{n+m}$ is defined by

$$K_{n,m}(f)(z) = (n+m+1) \sum_{k=0}^{n+m} \binom{n+m}{k} z^k (1-z)^{n+m-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt,$$

$z \in \mathbb{C}, \ f \in X_{D_R}$. We have

$$N(K_{n,m}) = \{f \in X_{D_R} : \ f(t) = 0, \ \ t \in D_R\}.$$

The operators $K_{n,m}$ are Hyers-Ulam stable since their ranges are finite dimensional spaces.

Theorem 2.11. *For* $n \ge 1$

$$K_{K_{n,m}} = \|\widetilde{T}_{n,m}^{-1}\| = \frac{(n+1)\binom{2(n+m)}{2[\frac{n+m}{2}]}}{(n+m+1)\binom{n+m}{[\frac{n+m}{2}]}}.$$

Proof. Let $p(z) \in \Pi_{n+m}$, $\|p\| \le 1$, and its Lorentz representation

$$p(z) = \sum_{k=0}^{n+m} c_k(p) z^k (1-z)^{n+m-k}, \ \ |z| \le R.$$

Consider the constant function $f_p \in X_{D_R}$ defined by

$$f_p(t) = \frac{(n+1)c_k(p)}{(n+m+1)\binom{n+m}{k}}, \ \ 0 \le k \le n+m, \ \ t \in D_R.$$

Then $K_{n,m}f_p = p$ and $\widetilde{K}_{n,m}^{-1}(p) = f_p + N(K_{n,m})$.

As usual, the norm of $\widetilde{K}_{n,m}^{-1} : \Pi_{n+m} \to X_{D_R}/N(K_{n,m})$ is defined by

$$\|\widetilde{K}_{n,m}^{-1}\| = \sup_{\|p\|\leq 1} \|\widetilde{K}_{n,m}^{-1}(p)\| = \sup_{\|p\|\leq 1} \inf_{h\in N(K_{n,m})} \|f_p + h\|. \quad \text{(by (2.5))}$$

Clearly

$$\inf_{h\in N(K_{n,m})} \|f_p + h\| = \|f_p\| = \max_{0\leq k\leq n+m} \frac{(n+1)|c_k(p)|}{(n+m+1)\binom{n+m}{k}}. \quad \text{(by (2.4))}$$

Therefore

$$\|\widetilde{K}_{n,m}^{-1}\| = \sup_{\|p\|\leq 1} \max_{0\leq k\leq n+m} \frac{(n+1)|c_k(p)|}{(n+m+1)\binom{n+m}{k}} \tag{2.9}$$

$$\leq \sup_{\|p\|\leq 1} \max_{0\leq k\leq n+m} \frac{(n+1)\|p\|d_{n+m,k}}{(n+m+1)\binom{n+m}{k}} \quad \text{(using Theorem (2.4))}$$

$$= \max_{0\leq k\leq n+m} \frac{(n+1)d_{n+m,k}}{(n+m+1)\binom{n+m}{k}}. \tag{2.10}$$

On the other hand, let $r(z) = T_n(2z-1)$, $|z| \leq R$. Then $\|r\| = 1$ and $|c_k(r)| = d_{n+m,k}$, $0 \leq k \leq n+m$, according to Theorem 2.4. Consequently by (2.9), we have

$$\|\widetilde{K}_{n,m}^{-1}\| \geq \max_{0\leq k\leq n+m} \frac{(n+1)|c_k(r)|}{(n+m+1)\binom{n+m}{k}} = \max_{0\leq k\leq n+m} \frac{(n+1)d_{n+m,k}}{(n+m+1)\binom{n+m}{k}} \tag{2.11}$$

and so by (2.10) and (2.11), we can conclude

$$\|\widetilde{K}_{n,m}^{-1}\| = \max_{0\leq k\leq n+m} \frac{(n+1)d_{n+m,k}}{(n+m+1)\binom{n+m}{k}} = \max_{0\leq k\leq n+m} \frac{(n+1)\binom{2(n+m)}{2k}}{(n+m+1)\binom{n+m}{k}}.$$

Let

$$a_k = \frac{(n+1)\binom{2(n+m)}{2k}}{(n+m+1)\binom{n+m}{k}}, \ 0 \leq k \leq n+m.$$

Then

$$\frac{a_{k+1}}{a_k} = \frac{2n+2m-2k-1}{2k+1}, \ 0 \leq k \leq n+m.$$

The inequality $\frac{a_{k+1}}{a_k} \geq 1$ is satisfied if and only if $k \leq [\frac{n+m-1}{2}]$, therefore

$$\max_{0\leq k\leq n+m} a_k = a_{[\frac{n+m-1}{2}]+1} = \begin{cases} a_{[\frac{n+m}{2}]}, & n+m \text{ is even;} \\ a_{[\frac{n+m}{2}]+1}, & n+m \text{ is odd.} \end{cases}$$

Since $a_{[\frac{n+m}{2}]+1} = a_{[\frac{n+m}{2}]}$ if $n+m$ is an odd number, we conclude that

$$K_{K_{n,m}} = \|\widetilde{L}_{n,m}^{-1}\| = \frac{(n+1)\binom{2(n+m)}{2[\frac{n+m}{2}]}}{(n+m+1)\binom{n+m}{[\frac{n+m}{2}]}}.$$

This completes the proof of the theorem. □

(iii) *Lorentz Operators*

The complex Lorentz polynomial [34] attached to any analytic function f in a domain containing the origin is given by

$$L_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{z}{n}\right)^k f^{(k)}(0), \quad n \in \mathbb{N}.$$

For $R > 1$ and denoting $D_R = \{z \in \mathbb{C}; |z| < R\}$, suppose that $f : D_R \to \mathbb{C}$ is analytic in D_R, i.e., $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$.

Theorem 2.12. *For each $n \geq 1$, the Lorentz polynomial on compact disk is not stable in the sense of Hyers and Ulam.*

Proof. To prove this theorem, we use the approach used in [89] in Theorem 4.1. Let us denote $e_j(z) = z^j$, then from Lorentz operators we can easily obtain that $L_n(e_0)(z) = 1$, $L_n(e_1)(z) = e_1(z)$ and that for all $j, n \in \mathbb{N},\ j \geq 2$, we have

$$\begin{aligned} L_n(e_j)(z) &= \binom{n}{j} j! \frac{z^j}{n^j}, \quad 1 \leq R_1 < R \\ &= z^j \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{j-1}{n}\right). \end{aligned}$$

Also, since an easy computation shows that

$$L_n(f)(z) = \sum_{j=0}^{\infty} c_j L_n(e_j)(z), \quad \forall |z| \leq R_1,$$

and $L_n(e_0)(z) = 1$, $L_n(e_1)(z) = e_1(z)$. It follows that for each $j \geq 2$, $(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{j-1}{n})$ is an eigen value of L_n. It can be easily seen that L_n is injective. Therefore $\frac{1}{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{j-1}{n})}$ is an eigen value of L_n^{-1}. Since

$$\lim_{j\to\infty} \frac{1}{(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{j-1}{n})} = \lim_{j\to\infty} \frac{n^j}{(n-1)(n-2)\cdots(n-j+1)} = +\infty,$$

we conclude that L_n^{-1} is unbounded and so L_n is not HU-stable. This completes the proof of the theorem. $\square$

Chapter 3

Approximation by Szász operators involving Brenke type polynomials

3.1 Introduction and preliminaries

In 2012, Varma et al. [108] constructed positive linear operators with the help of Brenke type polynomials. Brenke type polynomials [19] have generating functions of the form

$$A(t)B(xt)=\sum_{k=0}^{\infty}p_k(x)t^k \tag{3.1}$$

where A and B are analytic functions:

$$A(t)=\sum_{r=0}^{\infty}a_rt^r,\quad a_0\neq 0, \tag{3.2}$$

$$B(t)=\sum_{r=0}^{\infty}b_rt^r,\quad b_r\neq 0\ (r\geq 0) \tag{3.3}$$

and have the following explicit relation:

$$p_k(x)=\sum_{r=0}^{k}a_{k-r}b_rx^r,\quad k=0,1,2,\ldots. \tag{3.4}$$

Using the following restrictions:

(i) $A(1)\neq 0,\ \ \frac{a_{k-r}b_r}{A(1)}\geq 0,\ \ 0\leq r\leq k,\ \ k=0,1,2,\ldots,$

(ii) $B:[0,\infty)\to(0,\infty),$

(iii) (3.1) and the power series (3.2) and (3.3) converge for $|t|<R\ (R>1)$.

Varma et al [108] introduced the following positive linear operators involving the Brenke type polynomials

$$L_n(f;x):=\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right) \tag{3.5}$$

where $x \geq 0$ and $n \in \mathbb{N}$.

Let $B(t) = e^t$ and $A(t) = 1$. We meet the following Szász [104] operators

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \tag{3.6}$$

Motivated by the work of Büyükyazıcı [16], we consider the Chlodowsky [20] variant of Szász type operators involving the Brenke polynomials given by (3.5) as follows:

$$L_n^*(f;x) := \frac{1}{A(1)B\left(\frac{n}{b_n}x\right)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) f\left(\frac{k}{n}b_n\right) \tag{3.7}$$

where (b_n) is a positive increasing sequence such that

$$\lim_{n\to\infty} b_n = \infty, \quad \lim_{n\to\infty} \frac{b_n}{n} = 0 \tag{3.8}$$

and p_k are Brenke polynomials defined by (3.1).

The rest of the chapter is organized as follows. In Section 3.2 we obtain some local approximation results by the generalized Szász operators given by (3.7). In particular, the convergence of operators is examined with the help of Korovkin's theorem. The order of approximation is established by means of a classical approach, the second-order modulus of continuity and Peetre's K-functional. Section 3.3 is devoted to study some convergence properties of these operators in weighted spaces with weighted norm on the interval $[0,\infty)$ by using the weighted Korovkin-type theorems [30, 32]. Some examples are also given to compute error estimation by modulus of continuity in Section 3.4.

Note that throughout the paper we will assume that the operators L_n^* are positive and we use the following test functions

$$e_i(x) = x^i, \quad i \in \{0,1,2,3,4\}.$$

Also we consider

$$\lim_{y\to\infty} \frac{B^{(k)}(y)}{B(y)} = 1, \text{ for } k \in \{1,2,3,\ldots,r\}. \tag{3.9}$$

3.2 Local approximation properties of $L_n^*(f;x)$

We denote by $C_E[0,\infty)$ the set of all continuous functions f on $[0,\infty)$ with the property that $|f(x)| \leq \beta e^{\alpha x}$ for all $x \geq 0$ and some positive finite α and β. For a fixed $r \in \mathbb{N}$, we denote by $C_E^r[0,\infty) = \{f \in C_E[0,\infty) : f', f'', \cdots, f^{(r)} \in C_E[0,\infty)\}$. Using equality (3.1) and the fundamental properties of the L_n^* operators, one can easily get the following lemmas:

Lemma 3.1. *For all $x \in [0,\infty)$, we have*

$$L_n^*(e_0;x) = 1, \tag{3.10}$$

$$L_n^*(e_1;x) = \frac{B'\left(\frac{n}{b_n}x\right)}{B\left(\frac{n}{b_n}x\right)}x + \frac{b_n}{n}\frac{A'(1)}{A(1)}, \tag{3.11}$$

$$L_n^*(e_2;x) = \frac{B''\left(\frac{n}{b_n}x\right)}{B\left(\frac{n}{b_n}x\right)}x^2 + \frac{b_n}{n}\frac{\left(A(1)+2A'(1)\right)B'\left(\frac{n}{b_n}x\right)}{A(1)B\left(\frac{n}{b_n}x\right)}x + \frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}, \tag{3.12}$$

$$\begin{aligned} L_n^*(e_3;x) = {} & \frac{B'''\left(\frac{n}{b_n}x\right)}{B\left(\frac{n}{b_n}x\right)}x^3 + 3\frac{b_n}{n}\frac{\left(A(1)+A'(1)\right)B''\left(\frac{n}{b_n}x\right)}{A(1)B\left(\frac{n}{b_n}x\right)}x^2 \\ & + \frac{b_n^2}{n^2}\frac{\left(A(1)+6A'(1)+3A''(1)\right)B'\left(\frac{n}{b_n}x\right)}{A(1)B\left(\frac{n}{b_n}x\right)}x \\ & + \frac{b_n^3}{n^3}\frac{A'(1)+3A''(1)+A'''(1)}{A(1)}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} L_n^*(e_4;x) = {} & \frac{B^{(4)}\left(\frac{n}{b_n}x\right)}{B\left(\frac{n}{b_n}x\right)}x^4 + 2\frac{b_n}{n}\frac{\left(3A(1)+2A'(1)\right)B'''\left(\frac{n}{b_n}x\right)}{A(1)B\left(\frac{n}{b_n}x\right)}x^3 \\ & + \frac{b_n^2}{n^2}\frac{\left(7A(1)+18A'(1)+6A''(1)\right)B''\left(\frac{n}{b_n}x\right)}{A(1)B\left(\frac{n}{b_n}x\right)}x^2 \\ & + \frac{b_n^3}{n^3}\frac{\left(A(1)+14A'(1)+18A''(1)+4A'''(1)\right)B'\left(\frac{n}{b_n}x\right)}{A(1)B\left(\frac{n}{b_n}x\right)}x \\ & + \frac{b_n^4}{n^4}\frac{A'(1)+7A''(1)+6A'''(1)+A^{(4)}(1)}{A(1)}. \end{aligned} \tag{3.14}$$

Proof. From the generating functions of the Brenke type polynomials given by (3.1), we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) &= A(1)B\left(\frac{n}{b_n}x\right); \\ \sum_{k=0}^{\infty} kp_k\left(\frac{n}{b_n}x\right) &= A'(1)B\left(\frac{n}{b_n}x\right) + \frac{n}{b_n}xA(1)B'\left(\frac{n}{b_n}x\right); \\ \sum_{k=0}^{\infty} k^2p_k\left(\frac{n}{b_n}x\right) &= \frac{n^2}{b_n^2}x^2A(1)B''\left(\frac{n}{b_n}x\right) + \frac{n}{b_n}x\left(A(1)+2A'(1)\right)B'\left(\frac{n}{b_n}x\right) \\ &\quad + \left(A'(1)+A''(1)\right)B\left(\frac{n}{b_n}x\right); \end{aligned}$$

$$
\begin{aligned}
\sum_{k=0}^{\infty} k^3 p_k\Big(\frac{n}{b_n}x\Big) &= \frac{n^3}{b_n^3}x^3 A(1) B'''\Big(\frac{n}{b_n}x\Big) + 3\frac{n^2}{b_n^2}x^2\big(A(1)+A'(1)\big)B''\Big(\frac{n}{b_n}x\Big)\\
&+ \frac{n}{b_n}x\big(A(1)+6A'(1)+3A''(1)\big)B'\Big(\frac{n}{b_n}x\Big)\\
&+ \big(A'(1)+3A''(1)+A'''(1)\big)B\Big(\frac{n}{b_n}x\Big);\\
\sum_{k=0}^{\infty} k^4 p_k\Big(\frac{n}{b_n}x\Big) &= \frac{n^4}{b_n^4}x^4 A(1) B^{(4)}\Big(\frac{n}{b_n}x\Big) + 2\frac{n^3}{b_n^3}x^3\big(3A(1)+2A'(1)\big)B'''\Big(\frac{n}{b_n}x\Big)\\
&+ \frac{n^2}{b_n^2}x^2\big(7A(1)+18A'(1)+6A''(1)\big)B''\Big(\frac{n}{b_n}x\Big)\\
&+ \frac{n}{b_n}x\big(A(1)+14A'(1)+18A''(1)+4A'''(1)\big)B'\Big(\frac{n}{b_n}x\Big)\\
&+ \big(A'(1)+7A''(1)+6A'''(1)+A^{(4)}(1)\big)B\Big(\frac{n}{b_n}x\Big).
\end{aligned}
$$

In view of these equalities, we get our desired results.

It follows from Lemma 3.1 that,

$$
L_n^*\big((e_1-x);x\big) = \frac{B'\big(\frac{n}{b_n}x\big)-B\big(\frac{n}{b_n}x\big)}{B\big(\frac{n}{b_n}x\big)}x + \frac{b_n}{n}\frac{A'(1)}{A(1)}; \tag{3.15}
$$

$$
\begin{aligned}
L_n^*\big((e_1-x)^2;x\big) &= \frac{B''\big(\frac{n}{b_n}x\big)-2B'\big(\frac{n}{b_n}x\big)+B\big(\frac{n}{b_n}x\big)}{B\big(\frac{n}{b_n}x\big)}x^2\\
&+ \frac{b_n}{n}\frac{A(1)B'\big(\frac{n}{b_n}x\big)+2A'(1)\Big(B'\big(\frac{n}{b_n}x\big)-B\big(\frac{n}{b_n}x\big)\Big)}{A(1)B\big(\frac{n}{b_n}x\big)}x\\
&+ \frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)};
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
&L_n^*\big((e_1-x)^4;x\big)\\
&= \frac{B^{(4)}\big(\frac{n}{b_n}x\big)-4B'''\big(\frac{n}{b_n}x\big)+6B''\big(\frac{n}{b_n}x\big)-4B'\big(\frac{n}{b_n}x\big)+B\big(\frac{n}{b_n}x\big)}{B\big(\frac{n}{b_n}x\big)}x^4\\
&+ 2\frac{b_n}{n}\frac{1}{A(1)B\big(\frac{n}{b_n}x\big)}\bigg\{3A(1)\bigg(B'''\Big(\frac{n}{b_n}x\Big)-2B''\Big(\frac{n}{b_n}x\Big)+B'\Big(\frac{n}{b_n}x\Big)\bigg)\\
&+ 2A'(1)\bigg(B'''\Big(\frac{n}{b_n}x\Big)-3B''\Big(\frac{n}{b_n}x\Big)+3B'\Big(\frac{n}{b_n}x\Big)-B\Big(\frac{n}{b_n}x\Big)\bigg)\bigg\}x^3\\
&+ \frac{b_n^2}{n^2}\frac{1}{A(1)B\big(\frac{n}{b_n}x\big)}\bigg\{A(1)\bigg(7B''\Big(\frac{n}{b_n}x\Big)-4B'\Big(\frac{n}{b_n}x\Big)\bigg)+A'(1)\bigg(18B''\Big(\frac{n}{b_n}x\Big)\\
&- 24B'\Big(\frac{n}{b_n}x\Big)+6B\Big(\frac{n}{b_n}x\Big)\bigg)
\end{aligned}
$$

$$+ A''(1)\left(6B''\left(\frac{n}{b_n}x\right) - 12B'\left(\frac{n}{b_n}x\right) + 6B\left(\frac{n}{b_n}x\right)\right)\Big\}x^2$$
$$+ \frac{b_n^3}{n^3}\frac{1}{A(1)B(\frac{n}{b_n}x)}\Big\{A(1)B'\left(\frac{n}{b_n}x\right) + A'(1)\left(14B'\left(\frac{n}{b_n}x\right) - 4B\left(\frac{n}{b_n}x\right)\right)$$
$$+ A''(1)\left(18B'\left(\frac{n}{b_n}x\right) - 12B\left(\frac{n}{b_n}x\right)\right) + 4A'''(1)\left(B'\left(\frac{n}{b_n}x\right) - B\left(\frac{n}{b_n}x\right)\right)\Big\}x$$
$$+ \frac{b_n^4}{n^4}\frac{A'(1) + 7A''(1) + 6A'''(1) + A^{(4)}(1)}{A(1)}. \tag{3.17}$$

□

Theorem 3.2. *Let the condition (3.9) hold for $k = 1, 2$. Then for $f \in C_E[0,\infty)$, the operators L_n^* converge uniformly to f on $[0, a]$ as $n \to \infty$.*

Proof. According to (3.10)-(3.12), taking into account the equality (3.9), we find

$$\lim_{n\to\infty} L_n^*(e_i; x) = e_i(x), \quad i \in \{0, 1, 2\}.$$

If we apply the Korovkin theorem [2], we obtain the desired result. □

Example 3.3. *For (i) $A(t) = t$ and $B(t) = e^t$, (ii) $A(t) = e^{2t}$ and $B(t) = e^t$, the convergence of $L_n^*(f;x)$ to $f(x)$ is illustrated in Figs. 3.1 and 3.2, respectively, where $f(x) = xe^{-x^2}$, $n = 10,\ 100,\ 1000$, and $b_n = \sqrt{n}$.*

Now, we concerned with the estimate of the order of approximation of a function f by means of the positive operator L_n^*, using the first and second order modulus of continuity [26] given by (1.3) and (1.4), respectively.

Let $C_B[0,\infty)$ be the class of real valued functions defined on $[0,\infty)$ which are bounded and uniformly continuous with the norm $\|f\|_{C_B} = \sup_{x\in[0,\infty)} |f(x)|$ and $\omega,\ \omega_2$ be the first and second order moduli of continuity given by (1.3) and (1.4), respectively.

The Peetre's K-functional [26] of the function $f \in C_B[0,\infty)$ is defined by

$$K(f,\delta) := \inf_{g\in C_B^2[0,\infty)} \left\{\|f - g\|_{C_B} + \delta\|g\|_{C_B^2}\right\}$$

where

$$C_B^2[0,\infty) := \left\{g \in C_B[0,\infty) :\ g',\ g'' \in C_B[0,\infty)\right\}$$

and the norm $\|g\|_{C_B^2} := \|g\|_{C_B} + \|g'\|_{C_B} + \|g''\|_{C_B}$. It is clear that the inequality

$$K(f,\delta) \le M\left\{\omega_2(f,\sqrt{\delta}) + \min(1,\delta)\|f\|_{C_B}\right\}$$

is valid for all $\delta > 0$. The constant M is independent of f and δ.

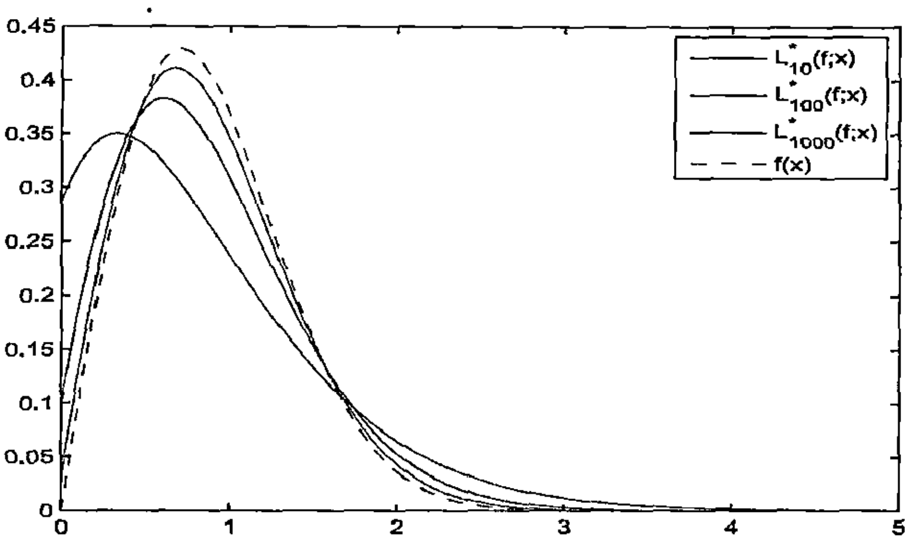

Fig.1. The convergence of $L^*_n(f;x)$ to f(x) taking A(t)=t & B(t)=e^t.

Figure 3.1:

Lemma 3.4. *([35]). Let $g \in C^2[0,\infty)$ and $(P_n)_{n\geq 0}$ be a sequence of positive linear operators with the property $P_n(1;x) = 1$. Then*

$$|P_n(g;x) - g(x)| \leq \|g'\|\sqrt{P_n\big((s-x)^2;x\big)} + \frac{1}{2}\|g''\|P_n\big((s-x)^2;x\big).$$

Lemma 3.5. *([111]). Let $f \in C[a,b]$ and $h \in \left(0, \frac{b-a}{2}\right)$. Let f_h be the second-order Steklov function attached to the function f. Then the following inequalities are satisfied:*

(i) $\|f_h - f\| \leq \frac{3}{4}\omega_2(f,h)$,

(ii) $\|f_h''\| \leq \frac{3}{2h^2}\omega_2(f,h)$.

Theorem 3.6. *If $f \in C_E[0,\infty)$, then for any $x \in [0,a]$ we have*

$$|L_n^*(f;x) - f(x)| \leq 2\omega\Big(f, \sqrt{\delta_n(a)}\Big)$$

where

$$\delta := \delta_n(a) = \frac{B''\left(\frac{n}{b_n}a\right) - 2B'\left(\frac{n}{b_n}a\right) + B\left(\frac{n}{b_n}a\right)}{B\left(\frac{n}{b_n}a\right)}a^2$$
$$+ \frac{b_n}{n}\frac{A(1)B'\left(\frac{n}{b_n}a\right) + 2A'(1)\Big(B'\left(\frac{n}{b_n}a\right) - B\left(\frac{n}{b_n}a\right)\Big)}{A(1)B\left(\frac{n}{b_n}a\right)}a + \frac{b_n^2}{n^2}\frac{A'(1) + A''(1)}{A(1)}.$$

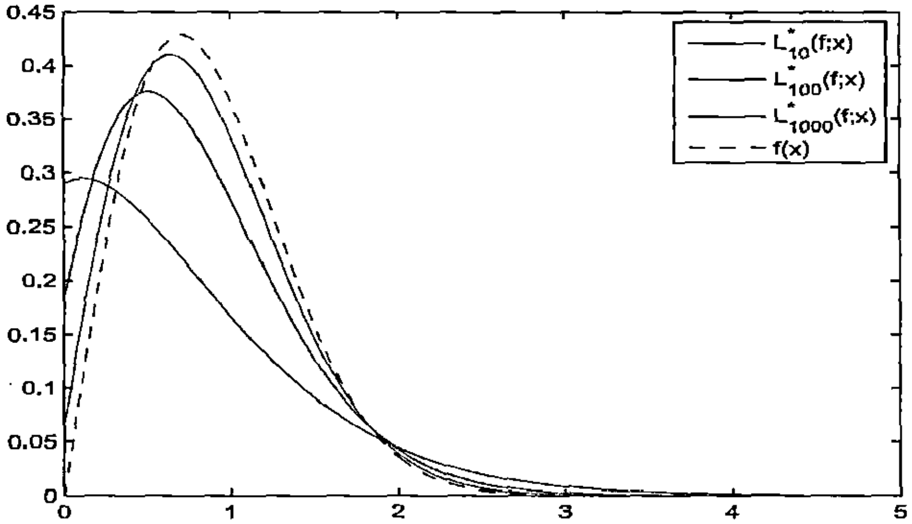

Fig.2. The convergence of $L^*_n(f;x)$ to $f(x)$ taking $A(t)=e^{2t}$ & $B(t)=e^t$.

Figure 3.2:

Proof. We will use the relation and the well-known properties of the modulus of continuity. We have

$$\begin{aligned}
&|L_n^*(f;x) - f(x)| \\
&\leq \frac{1}{A(1)B\left(\frac{n}{b_n}x\right)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)\left|f\left(\frac{k}{n}b_n\right) - f(x)\right| \\
&\leq \left\{1 + \frac{1}{\delta}\frac{1}{A(1)B\left(\frac{n}{b_n}x\right)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)\left|\frac{k}{n}b_n - x\right|\right\}\omega(f,\delta).
\end{aligned}$$

Using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
&|L_n^*(f;x) - f(x)| \\
&\leq \left\{1 + \frac{1}{\delta}\left(\frac{1}{A(1)B\left(\frac{n}{b_n}x\right)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)\left(\frac{k}{n}b_n - x\right)^2\right)^{\frac{1}{2}}\right\}\omega(f,\delta) \\
&= \left\{1 + \frac{1}{\delta}\sqrt{L_n^*\left((e_1 - x)^2;x\right)}\right\}\omega(f,\delta). \qquad (3.18)
\end{aligned}$$

By means of (3.16), for $0 \leq x \leq a$, one gets

$$L_n^*\left((e_1 - x)^2;x\right)$$

$$\leq \frac{B''\left(\frac{n}{b_n}a\right) - 2B'\left(\frac{n}{b_n}a\right) + B\left(\frac{n}{b_n}a\right)}{B\left(\frac{n}{b_n}a\right)} a^2$$
$$+ \frac{b_n}{n} \frac{A(1)B'\left(\frac{n}{b_n}a\right) + 2A'(1)\left(B'\left(\frac{n}{b_n}a\right) - B\left(\frac{n}{b_n}a\right)\right)}{A(1)B\left(\frac{n}{b_n}a\right)} a + \frac{b_n^2}{n^2} \frac{A'(1) + A''(1)}{A(1)}. \tag{3.19}$$

Using (3.19) and taking $\delta := \delta_n(a)$ in (3.18), we obtain the desired result. □

Theorem 3.7. *For $f \in C[0, a]$, the following inequality:*

$$|L_n^*(f; x) - f(x)| \leq \frac{2}{a}\|f\|h^2 + \frac{3}{4}(a + 2 + h^2)\omega_2(f, h)$$

is satisfied, where

$$h := h_n(x) = \sqrt[4]{L_n^*\left((e_1 - x)^2; x\right)}$$

and the second order modulus of continuity is given by $\omega_2(f, \delta)$ with the norm $\|f\| = \max\limits_{x \in [a,b]} |f(x)|$.

Proof. Let f_h be the second-order Steklov function attached to the function f. By virtue of the identity (3.10), we have

$$\begin{aligned} &|L_n^*(f; x) - f(x)| \\ &\leq |L_n^*(f - f_h; x)| + |L_n^*(f_h; x) - f_h(x)| + |f_h(x) - f(x)| \\ &\leq 2\|f_h - f\| + |L_n^*(f_h; x) - f_h(x)|. \end{aligned} \tag{3.20}$$

Taking into account the fact that $f_h \in C^2[0, a]$, it follows from Lemma 3.4 that

$$|L_n^*(f_h; x) - f_h(x)| \leq \|f_h'\| \sqrt{L_n^*\left((e_1 - x)^2; x\right)} + \frac{1}{2}\|f_h''\| L_n^*\left((e_1 - x)^2; x\right). \tag{3.21}$$

Combining the Landau inequality [62] and Lemma 3.5, we can write

$$\|f_h'\| \leq \frac{2}{a}\|f_h\| + \frac{a}{2}\|f_h''\| \leq \frac{2}{a}\|f\| + \frac{3a}{4}\frac{1}{h^2}\omega_2(f, h).$$

Now taking $h = \sqrt[4]{L_n^*\left((e_1 - x)^2; x\right)}$ in the above inequality, (3.21) becomes

$$|L_n^*(f_h; x) - f_h(x)| \leq \frac{2}{a}\|f\|h^2 + \frac{3a}{4}\omega_2(f, h) + \frac{3}{4}h^2\omega_2(f, h) \tag{3.22}$$

Substituting (3.22) in (3.21), and using Lemma 3.5, we get the desired result. □

Remark 3.8. *In Theorem 3.7, we give a proof for $h \in (0, \frac{a}{2})$. For the special case $B(t) = e^t$, $A(t) = 1$ and $x = 0$, one can deduce that $h = 0$ from the equality $h := h_n(x) = \sqrt[4]{L_n^*\big((e_1 - x)^2; x\big)}$. The inequality obtained in Theorem 3.7 still remains true when $h = 0$.*

Theorem 3.9. *Let $f \in C_B^2[0, \infty)$. Then*

$$|L_n^*(f; x) - f(x)| \leq \gamma_n(x) \|f\|_{C_B^2}$$

where

$$\begin{aligned}
\gamma(x) &:= \gamma_n(x) \\
&= \left\{ \frac{B''\big(\frac{n}{b_n}x\big) - 2B'\big(\frac{n}{b_n}x\big) + B\big(\frac{n}{b_n}x\big)}{B\big(\frac{n}{b_n}x\big)} x^2 \right. \\
&+ \left(\frac{\Big(B'\big(\frac{n}{b_n}x\big) - B\big(\frac{n}{b_n}x\big)\Big)\big(nA(1) + 2b_n A'(1)\big) + b_n A B'\big(\frac{n}{b_n}x\big)}{nA(1)B\big(\frac{n}{b_n}x\big)} \right) x \\
&+ \left. \frac{b_n^2}{n^2} \frac{A'(1) + A''(1)}{A(1)} + \frac{b_n}{n} \frac{A'(1)}{A(1)} \right\}.
\end{aligned}$$

Proof. Using the Taylor expansion of f, the linearity of the operators L_n^* and (3.10), it follows that

$$L_n^*(f; x) - f(x) = f'(x) L_n^*(e_1 - x; x) + \frac{1}{2} f''(\eta) L_n^*\big((e_1 - x)^2; x\big), \quad \eta \in (x, t). \quad (3.23)$$

Taking into account the fact that

$$L_n^*\big((e_1 - x); x\big) = \frac{B'\big(\frac{n}{b_n}x\big) - B\big(\frac{n}{b_n}x\big)}{B\big(\frac{n}{b_n}x\big)} x + \frac{b_n}{n} \frac{A'(1)}{A(1)} \geq 0$$

for $x \leq t$. By combining Lemma 3.1 and (3.16) in (3.23), we are led to

$$\begin{aligned}
&L_n^*(f; x) - f(x) \\
&\leq \left\{ \frac{B'\big(\frac{n}{b_n}x\big) - B\big(\frac{n}{b_n}x\big)}{B\big(\frac{n}{b_n}x\big)} x + \frac{b_n}{n} \frac{A'(1)}{A(1)} \right\} \|f'\|_{C_B} + \frac{1}{2} \left\{ \frac{B''\big(\frac{n}{b_n}x\big) - 2B'\big(\frac{n}{b_n}x\big) + B\big(\frac{n}{b_n}x\big)}{B\big(\frac{n}{b_n}x\big)} x^2 \right. \\
&+ \left. \frac{b_n}{n} \frac{A(1)B'\big(\frac{n}{b_n}x\big) + 2A'(1)\Big(B'\big(\frac{n}{b_n}x\big) - B\big(\frac{n}{b_n}x\big)\Big)}{A(1)B\big(\frac{n}{b_n}x\big)} x + \frac{b_n^2}{n^2} \frac{A'(1) + A''(1)}{A(1)} \right\} \|f''\|_{C_B} \\
&\leq \left\{ \frac{B''\big(\frac{n}{b_n}x\big) - 2B'\big(\frac{n}{b_n}x\big) + B\big(\frac{n}{b_n}x\big)}{B\big(\frac{n}{b_n}x\big)} x^2 \right.
\end{aligned}$$

$$+\left(\frac{(B'(\frac{n}{b_n}x)-B(\frac{n}{b_n}x))(nA(1)+2b_nA'(1))+b_nA(1)B'(\frac{n}{b_n}x)}{nA(1)B(\frac{n}{b_n}x)}\right)x$$
$$+\frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}+\frac{b_n}{n}\frac{A'(1)}{A(1)}\Bigg\}\|f\|_{C_B^2}$$

which completes the proof. □

Theorem 3.10. *Let $f\in C_B[0,\infty)$. Then*

$$|L_n^*(f;x)-f(x)|\le 2M\{\omega_2(f,\sqrt{\delta})+\min(1,\delta)\|f\|_{C_B}\}$$

where $\delta:=\delta_n(x)=\frac{1}{2}\gamma_n(x)$ and $M>0$ is a constant independent of the function f and δ. Note that $\gamma_n(x)$ is defined as in Theorem 3.9.

Proof. Let $g\in C_B^2[0,\infty)$. Theorem 3.9 allows us to write

$$\begin{aligned}
&|L_n^*(f;x)-f(x)|\\
&\le |L_n^*(f-g;x)|+|L_n^*(g;x)-g(x)|+|g(x)-f(x)|\\
&\le 2\|f-g\|_{C_B}+\left\{\frac{B''(\frac{n}{b_n}x)-2B'(\frac{n}{b_n}x)+B(\frac{n}{b_n}x)}{B(\frac{n}{b_n}x)}x^2\right.\\
&+\left(\frac{\left(B'(\frac{n}{b_n}x)-B(\frac{n}{b_n}x)\right)(nA(1)+2b_nA'(1))+b_nA(1)B'(\frac{n}{b_n}x)}{nA(1)B(\frac{n}{b_n}x)}\right)x\\
&\left.+\frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}+\frac{b_n}{n}\frac{A'(1)}{A(1)}\right\}\|g\|_{C_B^2}\\
&=2\{\|f-g\|_{C_B}+\delta\|g\|_{C_B^2}\}.
\end{aligned}\tag{3.24}$$

The left-hand side of inequality (3.24) does not depend on the function $g\in C_B^2[0,\infty)$, so

$$|L_n^*(f;x)-f(x)|\le 2K(f,\delta).\tag{3.25}$$

By using the relation between Peetre's K-functional and second modulus of smoothness, (3.25) becomes

$$|L_n^*(f;x)-f(x)|\le 2M\{\omega_2(f,\sqrt{\delta})+\min(1,\delta)\|f\|_{C_B}\}.$$

This completes the proof. □

Now we give a Voronoskaja-type relation for the operators (3.7).

Theorem 3.11. *If $f \in C_E^2[0,\infty)$, then*

$$\lim_{n\to\infty} \frac{n}{b_n}[L_n^*(f;x) - f(x)] = \frac{A'(1)}{A(1)} f'(x) + \frac{1}{2} x f''(x)$$

uniformly with respect to $x \in [0,a]$, provided the condition (3.9) is satisfied for $k \in \{1,2,3,4\}$.

Proof. For a fixed point $x_0 \in [0,\infty)$ and for all $x \in [0,\infty)$, by the Taylor formula we have

$$f(x) - f(x_0) = (x-x_0)f'(x_0) + \frac{1}{2}(x-x_0)^2 f''(x_0) + \varphi(x,x_0)(x-x_0)^2 \tag{3.26}$$

where $\varphi(x,x_0)$ is a function belonging to the space $C_E[0,\infty)$ and $\lim_{x\to x_0} \varphi(x,x_0) = 0$. By (3.10) and (3.26) we can write

$$\begin{aligned} \frac{n}{b_n}\left[L_n^*(f;x_0) - f(x_0)\right] &= \frac{n}{b_n} L_n^*\big((e_1 - x_0);x_0\big) f'(x_0) \\ &+ \frac{1}{2}\frac{n}{b_n} L_n^*\big((e_1-x_0)^2;x_0\big) f''(x_0) + \frac{n}{b_n} L_n^*\big(\varphi(t,x_0)(t-x_0)^2;x_0\big) \end{aligned} \tag{3.27}$$

for every $n \in \mathbb{N}$. Using (3.15), (3.16) and (3.9), we have

$$\lim_{n\to\infty} \frac{n}{b_n} L_n^*\big((e_1-x_0);x_0\big) = \frac{A'(1)}{A(1)}, \tag{3.28}$$

$$\lim_{n\to\infty} \frac{n}{b_n} L_n^*\big((e_1-x_0)^2;x_0\big) = x_0. \tag{3.29}$$

Applying the Cauchy-Schwartz inequality for the third term on the right hand side of (3.27), we get

$$\frac{n}{b_n} L_n^*\big(\varphi(t,x_0)(t-x_0)^2;x_0\big) \le \sqrt{\frac{n^2}{b_n^2} L_n^*\big((e_1-x_0)^4;x_0\big) L_n^*\big(\varphi^2(t,x_0);x_0\big)}. \tag{3.30}$$

One has from (3.17) and (3.9) that

$$\lim_{n\to\infty} \frac{n^2}{b_n^2} L_n^*\big((e_1-x_0)^4;x_0\big) = 3x_0^2. \tag{3.31}$$

Since for the function $\psi(x,x_0) = \varphi^2(x,x_0)$, $x \ge 0$, we have $\psi(x,x_0) \in C_E[0,\infty)$ and $\lim_{x\to x_0} \psi(x,x_0) = 0$, it follows from Theorem 3.2 that

$$\lim_{n\to\infty} L_n^*\big(\varphi^2(t,x_0);x_0\big) = \lim_{n\to\infty} L_n^*\big(\psi(t,x_0);x_0\big) = \psi(x_0,x_0) = 0 \tag{3.32}$$

uniformly with respect to $x_0 \in [0,a]$. So, considering (3.30)-(3.32), we obtain

$$\frac{n}{b_n} L_n^*\big(\varphi(t,x_0)(t-x_0)^2;x_0\big) = 0. \tag{3.33}$$

Now, taking the limit as $n \to \infty$ in (3.27) and using (3.28), (3.29) and (3.33) we have

$$\lim_{n\to\infty} \frac{n}{b_n}[L_n^*(f;x_0) - f(x_0)] = \frac{A'(1)}{A(1)} f'(x_0) + \frac{1}{2} x_0 f''(x_0).$$

The proof is completed. □

The following theorem gives the convergence of the derivatives of the L_n^* operators.

Theorem 3.12. *If $f \in C_E^r[0,\infty)$ and $\omega\left(\frac{d^r f}{dx^r},.\right)$ is the modulus of continuity of $\frac{d^r f}{dx^r}$, then taking into consideration the equality (3.9), for $k \in \{1,2,3,\ldots,r\}$, we have*

$$\frac{d^r}{dx^r} L_n^*(f;x) \to f^{(r)}(x) \quad as \quad n \to \infty.$$

Proof. By simple calculations, using the condition (3.9) the following formula is obtained

$$\lim_{n\to\infty} \frac{d^r}{dx^r} L_n^*(f;x) = \lim_{n\to\infty} \frac{1}{A(1)B\left(\frac{n}{b_n}x\right)} \left(\frac{n}{b_n}\right)^r \sum_{i=0}^{\infty} p_i\left(\frac{n}{b_n}x\right) \Delta_{\frac{b_n}{n}}^r f\left(\frac{i}{n}b_n\right)$$

where $\Delta_{\frac{b_n}{n}}^r f\left(\frac{i}{n}b_n\right)$ is the difference of order r of f corresponding to the increment $\frac{b_n}{n}$. Using the relation between finite difference and divided difference, the derivative of order r of the operator L_n^* is represented as follows:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{d^r}{dx^r} L_n^*(f;x) \\
&= \lim_{n\to\infty} \frac{r!}{A(1)B\left(\frac{n}{b_n}x\right)} \sum_{i=0}^{\infty} p_i\left(\frac{n}{b_n}x\right) \frac{\Delta_{\frac{b_n}{n}}^r f\left(\frac{i}{n}b_n\right)}{r!\left(\frac{b_n}{n}\right)^r} \\
&= \lim_{n\to\infty} \frac{r!}{A(1)B\left(\frac{n}{b_n}x\right)} \sum_{i=0}^{\infty} p_i\left(\frac{n}{b_n}x\right) \left[\frac{i}{n}b_n, \frac{i+1}{n}b_n, \ldots, \frac{i+r}{n}b_n; f\right] \\
&= \lim_{n\to\infty} \frac{r!}{A(1)B\left(\frac{n}{b_n}x\right)} \sum_{i=0}^{\infty} p_i\left(\frac{n}{b_n}x\right) \varphi\left(\frac{i}{n}b_n\right) \\
&= \lim_{n\to\infty} r! L_n^*(\varphi;x)
\end{aligned}$$

where $\varphi(x) = \left[x, x+\frac{b_n}{n}, \ldots, x+r\frac{b_n}{n}; f\right]$. Then, using Theorem 3.6, we have

$$\begin{aligned}
&\lim_{n\to\infty} \left|\frac{d^r}{dx^r} L_n^*(f;x) - f^{(r)}(x)\right| \\
&\leq \lim_{n\to\infty} r! \left|L_n^*(\varphi;x) - \varphi(x)\right| + \lim_{n\to\infty} \left|r!\varphi(x) - f^{(r)}(x)\right| \\
&\leq 2 \lim_{n\to\infty} r! \omega(f, \delta_n(a)) + \lim_{n\to\infty} \left|r!\varphi(x) - f^{(r)}(x)\right| \qquad (3.34)
\end{aligned}$$

where

$$\delta=\delta_n(a)=\frac{B''\left(\frac{n}{b_n}a\right)-2B'\left(\frac{n}{b_n}a\right)+B\left(\frac{n}{b_n}a\right)}{B\left(\frac{n}{b_n}a\right)}a^2$$
$$+\frac{b_n}{n}\frac{A(1)B'\left(\frac{n}{b_n}a\right)+2A'(1)\Big(B'\left(\frac{n}{b_n}a\right)-B\left(\frac{n}{b_n}a\right)\Big)}{A(1)B\left(\frac{n}{b_n}a\right)}a+\frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}.$$

We use the mean value theorem and some known classical properties of the modulus of continuity [2], and get

$$\begin{aligned}
&\left|\varphi(x+\delta)-\varphi(x)\right| \\
&=\left|\left[x+\delta,x+\delta+\frac{b_n}{n},\dots,x+\delta+r\frac{b_n}{n};f\right]-\left[x,x+\frac{b_n}{n},\dots,x+r\frac{b_n}{n};f\right]\right| \\
&=\frac{1}{r!}\left|\frac{d^r}{dx^r}f\left(x+\delta+r\frac{b_n}{n}\theta_1\right)-\frac{d^r}{dx^r}f\left(x+r\frac{b_n}{n}\theta_2\right)\right| \\
&\le\frac{1}{r!}\omega\left(f^{(r)},\delta+r\frac{b_n}{n}|\theta_1-\theta_2|\right) \\
&\le\frac{1}{r!}\omega\left(f^{(r)},\delta+r\frac{b_n}{n}\right)
\end{aligned}$$

where $\theta_1,\theta_2\in(0,1)$. Hence, we obtain

$$\omega(\varphi,\delta)\le\frac{1}{r!}\omega\left(f^{(r)},\delta+r\frac{b_n}{n}\right).$$

On the other hand

$$\begin{aligned}
&\left|r!\varphi(x)-f^{(r)}(x)\right| \\
&=\left|r!\left[x,x+\frac{b_n}{n},\dots,x+r\frac{b_n}{n};f\right]-f^{(r)}(x)\right| \\
&\le\left|\frac{d^r}{dx^r}f\left(x+r\frac{b_n}{n}\theta_3\right)-f^{(r)}(x)\right| \\
&\le\omega\left(f^{(r)},\theta_3r\frac{b_n}{n}\right) \\
&\le\omega\left(f^{(r)},r\frac{b_n}{n}\right)
\end{aligned}$$

where $\theta_3\in(0,1)$. Using the estimates in (3.34), we have

$$\lim_{n\to\infty}\left|\frac{d^r}{dx^r}L_n^*(f;x)-f^{(r)}(x)\right|\le 2r!\lim_{n\to\infty}\omega(f,\delta_n(a))+\lim_{n\to\infty}\omega\left(f^{(r)},r\frac{b_n}{n}\right).$$

Since $\delta_n(a)$ and $r\frac{b_n}{n}$ tend to zero as $n \to \infty$ imply that $\omega\big(f, \delta_n(a)\big)$ and $\omega\big(f^{(r)}, r\frac{b_n}{n}\big)$ tend to zero. Therefore from the last inequality we deduce

$$\frac{d^r}{dx^r} L_n^*(f;x) \to f^{(r)}(x) \quad \text{as} \quad n \to \infty.$$

This completes the proof. $\square$

Example 3.13. *For (i) $A(t) = t$ and $B(t) = e^t$, (ii) $A(t) = e^{2t}$ and $B(t) = e^t$, the convergence of $\frac{d}{dx}L_n^*(f;x)$ to $f'(x)$ is illustrated in Figs. 3.3 and 3.4, respectively, where $f(x) = xe^{-x^2}$, $n = 10,\ 100,\ 1000$, and $b_n = \sqrt{n}$, $r = 1$.*

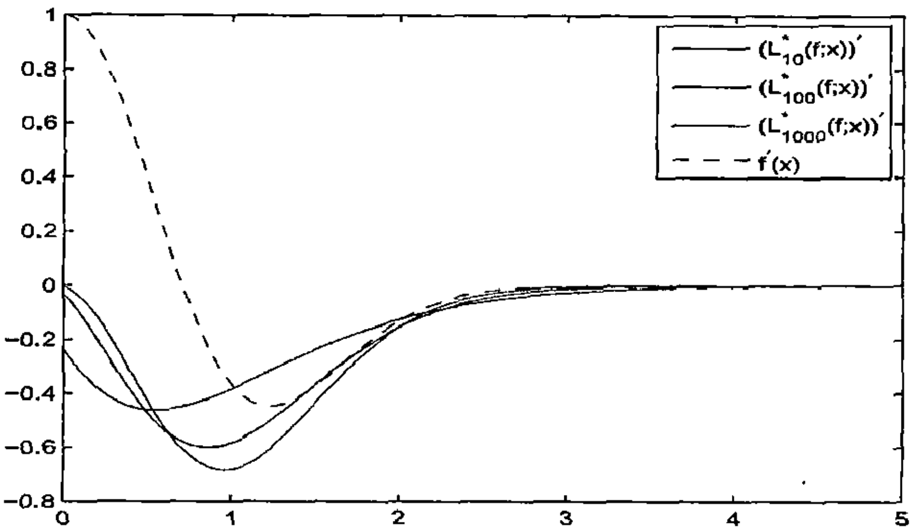

Fig.3. The convergence of $(L_n^*(f;x))'$ to $f'(x)$ taking $A(t)=t$ & $B(t)=e^t$.

Figure 3.3:

3.3 Approximation properties in weighted spaces

Now we give approximation properties of the operators L_n^* of the weighted spaces of continuous functions with exponential growth on $\mathbb{R}_0^+ = [0, \infty)$ with the help of the weighted Korovkin type theorem proved by Gadjiev in [30, 32]. For this purpose, we consider the following weighted spaces of functions which are defined on the $\mathbb{R}_0^+ = [0, \infty)$.

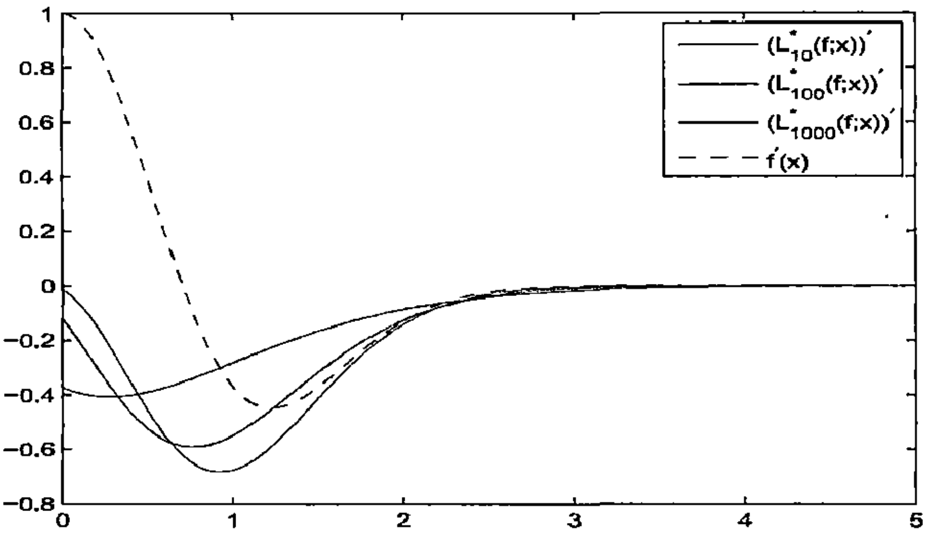

Fig.4. The convergence of $(L^*_n(f;x))'$ to $f'(x)$ taking $A(t)=e^{2t}$ & $B(t)=e^t$.

Figure 3.4:

Let $\rho(x)$ be the weight function and M_f be a positive constant. Then we define

$$\begin{aligned} B_\rho(\mathbb{R}_0^+) &= \{f \in E(\mathbb{R}_0^+) : |f(x) \leq M_f \rho(x)|\}, \\ C_\rho(\mathbb{R}_0^+) &= \{f \in B_\rho(\mathbb{R}_0^+) : f \text{ is continuous}\}, \\ C_\rho^k(\mathbb{R}_0^+) &= \left\{f \in C_\rho(\mathbb{R}_0^+) : \lim_{n\to\infty} \frac{f(x)}{\rho(x)} = K_f < \infty\right\}. \end{aligned}$$

It is obvious that $C_\rho^k(\mathbb{R}_0^+) \subset C_\rho(\mathbb{R}_0^+) \subset B_\rho(\mathbb{R}_0^+)$. The space $B_\rho(\mathbb{R}_0^+)$ is a normed linear space with the following norm:

$$\|f\|_\rho = \sup_{x\in\mathbb{R}_0^+} \frac{|f(x)|}{\rho(x)}.$$

The following results on the sequence of positive linear operators in these spaces are given in [30, 32].

Lemma 3.14. *([30, 32]) The sequence of positive linear operators $(L_n)_{n\geq 1}$ acts from $C_\rho(\mathbb{R}_0^+)$ to $B_\rho(\mathbb{R}_0^+)$ if and only if there exists a positive constant k such that*

$$L_n(\rho; x) \leq k\rho(x), \quad \textit{i.e.}$$

$$\|L_n(\rho; x)\|_\rho \leq k.$$

Theorem 3.15. *([30, 32]) Let $(L_n)_{n\geq 1}$ be the sequence of positive linear operators which act from $C_\rho(\mathbb{R}_0^+)$ to $B_\rho(\mathbb{R}_0^+)$ such that*

$$\lim_{n\to\infty} \|L_n(t^i;x) - x^i\|_\rho = 0, \quad i \in \{0,1,2\}.$$

Then for any function $f \in C_\rho^k(\mathbb{R}_0^+)$

$$\lim_{n\to\infty} \|L_n f - f\|_\rho = 0.$$

Lemma 3.16. *Let $\rho(x) = 1 + x^2$ be a weight function. If $f \in C_\rho(\mathbb{R}_0^+)$, then*

$$\|L_n^*(\rho;x)\|_\rho \leq 1 + M$$

under the equality (3.9) for $k = 1,2$.

Proof. Using (3.10) and (3.12), one has

$$L_n^*(\rho;x) = 1 + \frac{B''\left(\frac{n}{b_n}x\right)}{B\left(\frac{n}{b_n}x\right)}x^2 + \frac{b_n}{n}\frac{\left(A(1)+2A'(1)\right)B'\left(\frac{n}{b_n}x\right)}{A(1)B\left(\frac{n}{b_n}x\right)}x + \frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)},$$

$$\begin{aligned}
&\|L_n^*(\rho;x)\|_\rho \\
&= \sup_{x\geq 0}\left\{\frac{1}{1+x^2}\left(1 + \frac{B''\left(\frac{n}{b_n}x\right)}{B\left(\frac{n}{b_n}x\right)}x^2 + \frac{b_n}{n}\frac{\left(A(1)+2A'(1)\right)B'\left(\frac{n}{b_n}x\right)}{A(1)B\left(\frac{n}{b_n}x\right)}x + \frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}\right)\right\} \\
&\leq 1 + \frac{B''\left(\frac{n}{b_n}x\right)}{B\left(\frac{n}{b_n}x\right)} + \frac{b_n}{n}\frac{\left(A(1)+2A'(1)\right)B'\left(\frac{n}{b_n}x\right)}{A(1)B\left(\frac{n}{b_n}x\right)} + \frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}.
\end{aligned}$$

Since $\lim_{n\to\infty}\frac{b_n}{n} = 0$ and using condition (3.9), there exists a positive M such that

$$\|L_n^*(\rho;x)\|_\rho \leq 1 + M.$$

This completes the proof. □

By using Lemma 3.16, we can easily see that the operators L_n^* defined by (3.7) act from $C_\rho(\mathbb{R}_0^+)$ to $B_\rho(\mathbb{R}_0^+)$.

Theorem 3.17. *Let L_n^* be the sequence of positive linear operators defined by (3.7) and $\rho(x) = 1 + x^2$, then for each $f \in C_\rho^k(\mathbb{R}_0^+)$*

$$\lim_{n\to\infty} \|L_n^*(f;x) - f(x)\|_\rho = 0$$

provided the equality (3.9) holds for $k = 1,2$.

Proof. It is enough to prove that the conditions of the weighted Korovkin type theorem given by Theorem 3.15 are satisfied. From (3.10), it is immediate that

$$\lim_{n\to\infty} \|L_n^*(e_0;x) - e_0(x)\|_\rho = 0. \tag{3.35}$$

Using (3.11) and the condition (3.9), we have

$$\|L_n^*(e_1;x) - e_1(x)\|_\rho = \frac{b_n}{n}\frac{A'(1)}{(1)} \tag{3.36}$$

which implies that

$$\lim_{n\to\infty} \|L_n^*(e_1;x) - e_1(x)\|_\rho = 0. \tag{3.37}$$

By means of (3.12), we get

$$\begin{aligned}
&\|L_n^*(e_2;x) - e_2(x)\|_\rho \\
&= \sup_{x\in R_0}\left| \frac{B''\left(\frac{n}{b_n}x\right)}{B\left(\frac{n}{b_n}x\right)}\frac{x^2}{1+x^2} + \frac{b_n}{n}\frac{\left(A(1)+2A'(1)\right)B'\left(\frac{n}{b_n}x\right)}{A(1)B\left(\frac{n}{b_n}x\right)}\frac{x}{1+x^2} \right. \\
&\left. + \frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}\frac{1}{1+x^2}\right| \\
&\le \frac{B''\left(\frac{n}{b_n}x\right)}{B\left(\frac{n}{b_n}x\right)} + \frac{b_n}{n}\frac{\left(A(1)+2A'(1)\right)B'\left(\frac{n}{b_n}x\right)}{A(1)B\left(\frac{n}{b_n}x\right)} + \frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}.
\end{aligned} \tag{3.38}$$

Using the conditions (3.4) and (3.9), it follows that

$$\lim_{n\to\infty} \|L_n^*(e_2;x) - e_2(x)\|_\rho = 0. \tag{3.39}$$

From (3.35), (3.36) and (3.39), for $i \in \{0,1,2\}$, we have

$$\lim_{n\to\infty} \|L_n^*(t^i;x) - x^i\|_\rho = 0.$$

Applying Theorem 3.15, we obtain the desired result. $\square$

3.4 Numerical Examples

Example 3.18. *The error bound for the function $f(x) = xe^{-x^2}$ under different conditions is computed in the following table:*

n	*Error estimate by L_n^* operators taking $A(t) = t$ and $B(t) = e^t$*	*Error estimate by L_n^* operators taking $A(t) = e^{2t}$ and $B(t) = e^t$*
10	0.3747901819	0.8372115913
10^2	0.1176870475	0.1580928811
10^3	0.0367765764	0.0409630153
10^4	0.0115759974	0.0119975423
10^5	0.0036549045	0.0036972316
10^6	0.0011551773	0.0011594048
10^7	3.6524581199e-004	3.6566718205e-004
10^8	1.1549548964e-004	1.1553763604e-004
10^9	3.6522341512e-005	3.6526556448e-005
10^{10}	1.1549324506e-005	1.1549746001e-005

Table 3.1: The error bound of function $f(x) = xe^{-x^2}$ by using modulus of continuity

Chapter 4

Some results on the approximation by q-Beta operators

4.1 Introduction and preliminaries

In [100], Stancu introduced the sequence of Beta operators of second kind in order to approximate the Lebesgue integrable functions on the interval $(0,\infty)$. Motivated by the q-calculus which has been one of the most interesting areas of research in the last decade, Ali and Gupta [6] constructed the q-analogue of the Stancu-Beta operators and they established direct results in terms of modulus of continuity and also presented an asymptotic formula for Voronovskaja type theorem. Mursaleen et al. [77] gave another generalization of the operators defined by Ali and Gupta [6] and they studied statistical approximation properties of the operators.

Let $B_m[0,\infty)$ be the set of all functions f satisfying the condition that $|f(x)| \leq M_f(1+x^m)$, $x \in [0,\infty)$, $m > 0$ with some constant M_f depending on f. Introduce

$$C_m[0,\infty) := B_m[0,\infty) \cap C[0,\infty),$$

$$C_m^*[0,\infty) := \left\{ f \in C_m[0,\infty) : \exists \lim_{x\to\infty} \frac{|f(x)|}{1+x^m} < \infty \right\}.$$

These spaces are endowed with the norm

$$\|f\|_m := \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^m}.$$

The aim of this chapter is to introduce a two parametric q-analogue of Stancu-Beta operators and establish some direct results in the polynomial weighted space of continuous functions defined on the interval $[0,\infty)$. Then we obtain point-wise estimate using the Lipschitz type maximal function. Furthermore, we obtain a Voronovskaja type theorem for these operators. Moreover, we also establish some results for q-Stancu-Beta operators which preserve x^2, the operators defined in [17].

4.2 Two parametric q-Stancu-Beta operators

In order to introduce two parametric q-Stancu-Beta operators, we present a construction due to Aral and Gupta [6].

Definition 4.1. *Let $q \in (0,1)$ and $n \in \mathbb{N}$. For $f : [0,\infty) \to \mathbb{R}$, the q-analogue of Stancu-Beta operators are defined as*

$$L_n^q(f;x) = \frac{K(A,[n]_q x)}{B_q([n]_q x, [n]_q + 1)} \int_0^{\infty/A} \frac{u^{[n]_q x - 1}}{(1+u)_q^{[n]_q x + [n]_q + 1}} f\left(q^{[n]_q x} u\right) d_q u. \tag{4.1}$$

Lemma 4.2. *([6]) We have*

$$L_n^q(e_0;x) = 1, \quad L_n^q(e_1;x) = x, \quad L_n^q(e_2;x) = \frac{([n]_q x + 1)x}{q([n]_q - 1)}.$$

Now we propose the q-analogue of Stancu-Beta operators with two parameters α and β.

Definition 4.3. *Let $q \in (0,1)$ and $n \in \mathbb{N}$. For $f : [0,\infty) \to \mathbb{R}$, the q-analogue of Stancu-Beta operators are defined as*

$$\begin{aligned} &L_{n,q}^{(\alpha,\beta)}(f;x) \\ &= \frac{K(A,[n]_q x)}{B_q([n]_q x, [n]_q + 1)} \int_0^{\infty/A} \frac{u^{[n]_q x - 1}}{(1+u)_q^{[n]_q x + [n]_q + 1}} f\left(\frac{[n]_q q^{[n]_q x} u + \alpha}{[n]_q + \beta}\right) d_q u. \end{aligned} \tag{4.2}$$

where $\alpha \leq \beta$. If we take $\alpha = \beta = 0$ in the above operator it reduces to the operator (4.1).

Moments $L_{n,q}^{(\alpha,\beta)}(e_m;x)$ are of particular importance in approximation theory by positive operators. From (4.2) we easily derive the following formula for moments $L_{n,q}^{(\alpha,\beta)}(e_m;x)$, $m = 0,1,2$.

Lemma 4.4. *The operators defined at (4.2) verify the following identities*

$$L_{n,q}^{(\alpha,\beta)}(e_0;x) = 1, \quad L_{n,q}^{(\alpha,\beta)}(e_1;x) = \frac{[n]_q}{[n]_q + \beta} x + \frac{\alpha}{[n]_q + \beta},$$

$$L_{n,q}^{(\alpha,\beta)}(e_2;x) = \frac{[n]_q^2([n]_q x + 1)x}{q([n]_q + \beta)^2([n]_q - 1)} + \frac{2\alpha [n]_q x}{([n]_q + \beta)^2} + \frac{\alpha^2}{([n]_q + \beta)^2}.$$

Proof. By the definition of q-Stancu-Beta operators we have

$$\begin{aligned}
L_{n,q}^{(\alpha,\beta)}(e_0;x) &= \frac{K(A,[n]_qx)}{B_q([n]_qx,[n]_q+1)}\int_0^{\infty/A}\frac{u^{[n]_qx-1}}{(1+u)_q^{[n]_qx+[n]_q+1}}\mathrm{d}_qu \\
&= L_n^q(e_0;x) = 1.
\end{aligned}$$

$$\begin{aligned}
L_{n,q}^{(\alpha,\beta)}(e_1;x) &= \frac{K(A,[n]_qx)}{B_q([n]_qx,[n]_q+1)}\int_0^{\infty/A}\frac{u^{[n]_qx-1}}{(1+u)_q^{[n]_qx+[n]_q+1}}\frac{[n]_qq^{[n]_qx}u+\alpha}{[n]_q+\beta}\mathrm{d}_qu \\
&= \frac{[n]_q}{[n]_q+\beta}\frac{K(A,[n]_qx)q^{[n]_qx}}{B_q([n]_qx,[n]_q+1)}\int_0^{\infty/A}\frac{u^{[n]_qx}}{(1+u)_q^{[n]_qx+[n]_q+1}}\mathrm{d}_qu \\
&+ \frac{\alpha}{[n]_q+\beta}\frac{K(A,[n]_qx)}{B_q([n]_qx,[n]_q+1)}\int_0^{\infty/A}\frac{u^{[n]_qx-1}}{(1+u)_q^{[n]_qx+[n]_q+1}}\mathrm{d}_qu \\
&= \frac{[n]_q}{[n]_q+\beta}L_n^q(e_1;x)+\frac{\alpha}{[n]_q+\beta}L_n^q(e_0;x) \\
&= \frac{[n]_q}{[n]_q+\beta}x+\frac{\alpha}{[n]_q+\beta}.
\end{aligned}$$

Finally, by using the q-Beta integral, we have

$$\begin{aligned}
L_{n,q}^{(\alpha,\beta)}(e_2;x) &= \frac{K(A,[n]_qx)}{B_q([n]_qx,[n]_q+1)}\int_0^{\infty/A}\frac{u^{[n]_qx-1}}{(1+u)_q^{[n]_qx+[n]_q+1}}\left(\frac{[n]_qq^{[n]_qx}u+\alpha}{[n]_q+\beta}\right)^2\mathrm{d}_qu \\
&= \frac{[n]_q^2}{([n]_q+\beta)^2}\frac{K(A,[n]_qx)q^{2[n]_qx}}{B_q([n]_qx,[n]_q+1)}\int_0^{\infty/A}\frac{u^{[n]_qx+1}}{(1+u)_q^{[n]_qx+[n]_q+1}}\mathrm{d}_qu \\
&+ \frac{2\alpha[n]_q}{([n]_q+\beta)^2}\frac{K(A,[n]_qx)q^{[n]_qx}}{B_q([n]_qx,[n]_q+1)}\int_0^{\infty/A}\frac{u^{[n]_qx}}{(1+u)_q^{[n]_qx+[n]_q+1}}\mathrm{d}_qu \\
&+ \frac{\alpha^2}{([n]_q+\beta)^2}\frac{K(A,[n]_qx)}{B_q([n]_qx,[n]_q+1)}\int_0^{\infty/A}\frac{u^{[n]_qx-1}}{(1+u)_q^{[n]_qx+[n]_q+1}}\mathrm{d}_qu \\
&= \frac{[n]_q^2}{([n]_q+\beta)^2}L_n^q(e_2;x)+\frac{2\alpha[n]_q}{([n]_q+\beta)^2}L_n^q(e_1;x)+\frac{\alpha^2}{([n]_q+\beta)^2}L_n^q(e_0;x) \\
&= \frac{[n]_q^2([n]_qx+1)x}{q([n]_q+\beta)^2([n]_q-1)}+\frac{2\alpha[n]_qx}{([n]_q+\beta)^2}+\frac{\alpha^2}{([n]_q+\beta)^2}.
\end{aligned}$$

□

Remark 4.5. *Suppose that $q \in (0,1)$. Then for $x \in [0,\infty)$, we can have the following formula for the m-th order moment:*

$$L_{n,q}^{(\alpha,\beta)}(e_m;x) = \sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^{m-j}\alpha^j}{([n]_q+\beta)^m}\frac{\Gamma_q\big([n]_qx+m-j\big)\Gamma_q\big([n]_q-m+j+1\big)}{\Gamma_q\big([n]_qx\big)\Gamma_q\big([n]_q+1\big)q^{(m-j)(m-j-1)/2}}.$$

Lemma 4.6. *Let $n > 1$ be a given number. For every $q \in (0,1)$, we have*

$$L_{n,q}^{(\alpha,\beta)}((e_1 - e_0x)^2; x)$$
$$\le \left(\frac{[n]_q}{q([n]_q-1)} - \frac{[n]_q-\beta}{[n]_q+\beta}\right)x^2 + \frac{x}{q([n]_q-1)} + \frac{\alpha^2}{([n]_q+\beta)^2}$$
$$\le \frac{2(\beta+1)^2x^2 + x + \alpha^2}{q([n]_q-1)}$$

where $x \in [0,\infty)$.

Proof. By using the linearity of the operator, we have

$$L_{n,q}^{(\alpha,\beta)}((e_1 - e_0x)^2; x)$$
$$= \frac{[n]_q^2([n]_qx+1)x}{q([n]_q+\beta)^2([n]_q-1)} + \frac{2\alpha[n]_qx}{([n]_q+\beta)^2} + \frac{\alpha^2}{([n]_q+\beta)^2} - 2x\left\{\frac{[n]_q}{[n]_q+\beta}x + \frac{\alpha}{[n]_q+\beta}\right\} + x^2$$
$$= \left\{\frac{[n]_q^3}{q([n]_q+\beta)^2([n]_q-1)} - \frac{2[n]_q}{[n]_q+\beta} + 1\right\}x^2$$
$$+ \left\{\frac{[n]_q^2}{q([n]_q+\beta)^2([n]_q-1)} + \frac{2\alpha[n]_q}{([n]_q+\beta)^2} - \frac{2\alpha}{[n]_q+\beta}\right\}x + \frac{\alpha^2}{([n]_q+\beta)^2}$$
$$\le \left(\frac{[n]_q}{q([n]_q-1)} - \frac{[n]_q-\beta}{[n]_q+\beta}\right)x^2 + \frac{x}{q([n]_q-1)} + \frac{\alpha^2}{([n]_q+\beta)^2}$$
$$= \frac{\{(1-q)[n]_q^3 + ([n]_q + q[n]_q - q)\beta^2 + (2\beta+q)[n]_q^2\}x^2 + ([n]_q+\beta)^2x + q([n]_q-1)\alpha^2}{q([n]_q-1)([n]_q+\beta)^2}$$
$$= \frac{\{(1-q^n)[n]_q^2 + ([n]_q + q[n]_q - q)\beta^2 + (2\beta+q)[n]_q^2\}x^2 + ([n]_q+\beta)^2x + q([n]_q-1)\alpha^2}{q([n]_q-1)([n]_q+\beta)^2}$$
$$\le \frac{\{[n]_q^2 + (1+q)[n]_q^2\beta^2 + (2\beta+q)[n]_q^2\}x^2 + ([n]_q+\beta)^2x + [n]_q\alpha^2}{q([n]_q-1)([n]_q+\beta2)^2}$$
$$\le \frac{2(\beta^2+\beta+1)x^2 + x + \alpha^2}{q([n]_q-1)}$$
$$\le \frac{2(\beta+1)^2x^2 + x + \alpha^2}{q([n]_q-1)}.$$

□

4.2.1 Rate of approximation

Let $H_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)| \le M_f(1+x^2)$, where M_f is a constant depending only on f. By $C_{x^2}[0,\infty)$, we denote the subspace of all continuous functions belonging to $H_{x^2}[0,\infty)$. Also, let $C_{x^2}^*[0,\infty)$

be the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{|x|\to\infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}$. We denote the modulus of continuity of f on the closed interval $[0,a]$, $a>0$ by

$$\omega_a(f,\delta) = \sup_{|t-x|\le\delta} \sup_{x,t\in[0,a]} |f(t)-f(x)|.$$

We observe that for $f \in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero.

Now we give a rate of convergence theorem for the operator $L_{n,q}^{(\alpha,\beta)}(f;x)$.

Theorem 4.7. *Let $f \in C_{x^2}[0,\infty)$, $q\in(0,1)$ and $\omega_{a+1}(f,\delta)$ be the modulus of continuity on the finite interval $[0,a+1]\subset[0,\infty)$, where $a>0$. Then for every $n\ge 2$*

$$\begin{aligned}&\left\|L_{n,q}^{(\alpha,\beta)}(f)-f\right\|_{C[0,a]}\\ &\le \frac{4M_f(1+a^2)\left(2(\beta+1)^2a^2+\alpha^2+a\right)}{q([n]_q-1)} + 2\omega_{a+1}\left(f,\sqrt{\frac{2(\beta+1)^2a^2+\alpha^2+a}{q([n]_q-1)}}\right).\end{aligned}$$

Proof. For $x\in[0,a]$ and $t>a+1$, since $t-x>1$, we have

$$\begin{aligned}|f(t)-f(x)| &\le M_f(2+x^2+t^2)\le M_f\left(2+3x^2+2(t-x)^2\right)\\ &\le M_f(4+3x^2)(t-x)^2 \le 4M_f(1+a^2)(t-x)^2.\end{aligned}\tag{4.3}$$

For $x\in[0,a]$ and $t\le a+1$, we have

$$|f(t)-f(x)|\le\omega_{a+1}(f,|t-x|)\le\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta)\tag{4.4}$$

with $\delta>0$. From (4.3) and (4.4), we may write

$$|f(t)-f(x)|\le 4M_f(1+a^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta),\tag{4.5}$$

for $x\in[0,a]$ and $t\ge0$. Thus

$$\begin{aligned}\left|L_{n,q}^{(\alpha,\beta)}(f;x)-f(x)\right| &\le L_{n,q}^{(\alpha,\beta)}(|f(t)-f(x)|;x)\\ &\le 4M_f(1+a^2)L_{n,q}^{(\alpha,\beta)}\left((t-x)^2;x\right)+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{L_{n,q}^{(\alpha,\beta)}\left((t-x)^2;x\right)}\right).\end{aligned}$$

Hence, by Schwartz's inequality and Lemma 4.6, for every $q\in(0,1)$ and $x\in[0,a]$,

$$\left|L_{n,q}^{(\alpha,\beta)}(f;x)-f(x)\right|$$

$$\leq 4M_f(1+a^2)\left(\frac{2(\beta+1)^2x^2+x+\alpha^2}{q([n]_q-1)}\right)+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{2(\beta+1)^2x^2+x+\alpha^2}{q([n]_q-1)}}\right)$$

$$\leq \frac{4M_f(1+a^2)(2(\beta+1)^2a^2+\alpha^2+a)}{q([n]_q-1)}+\omega_{a+1}\left(1+\frac{1}{\delta}\sqrt{\frac{2(\beta+1)^2a^2+\alpha^2+a}{q([n]_q-1)}}\right).$$

By taking $\delta=\sqrt{\frac{2(\beta+1)^2a^2+\alpha^2+a}{q([n]_q-1)}}$, we get the desired result. □

In approximation theory, weighted approximation for certain family of summation integral type operators are studied in [110]. Our next result in this section is to discuss the weighted approximation result for the operators $L_{n,q}^{(\alpha,\beta)}(f;x)$, where the approximation formula holds true on $[0,\infty)$.

Theorem 4.8. *Let $q=q_n$ satisfies $0<q_n<1$ and let $q_n\to 1$ as $n\to\infty$. For each $f\in C^*_{x^2}[0,\infty)$, we have*

$$\lim_{n\to\infty}\|L_{n,q_n}^{(\alpha,\beta)}(f)-f\|_{x^2}=0.$$

Proof. Using the theorem in [32] we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\|L_{n,q_n}^{(\alpha,\beta)}(e_m;x)-e_m\|_{x^2}=0,\quad m=0,1,2. \tag{4.6}$$

Since $L_{n,q_n}^{(\alpha,\beta)}(e_0;x)=1$, (4.6) holds true for $m=0$. By Lemma 4.4, we have

$$\|L_{n,q_n}^{(\alpha,\beta)}(e_1;x)-e_1\|_{x^2}=\sup_{x\in[0,\infty)}\frac{|L_{n,q_n}^{(\alpha,\beta)}(t;x)-x|}{1+x^2}=\sup_{x\in[0,\infty)}\frac{1}{1+x^2}\left|\frac{[n]_{q_n}x+\alpha}{[n]_{q_n}+\beta}-x\right|$$

$$\leq\frac{\beta}{[n]_{q_n}+\beta}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{\alpha}{[n]_{q_n}+\beta}\leq\frac{\beta}{[n]_{q_n}+\beta}+\frac{\alpha}{[n]_{q_n}+\beta}.$$

In the case $m=1$, (4.6) is also true for $n\to\infty$.

Similarly we can write

$$\|L_{n,q_n}^{(\alpha,\beta)}(e_2;x)-e_2\|_{x^2}=\sup_{x\in[0,\infty)}\frac{|L_{n,q_n}^{(\alpha,\beta)}(t^2;x)-x^2|}{1+x^2}$$

$$\leq\left(\frac{[n]_{q_n}^3}{q_n([n]_{q_n}-1)([n]_{q_n}+\beta)^2}-1\right)\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}$$

$$+\left(\frac{[n]_{q_n}^2+2q_n[n]_{q_n}([n]_{q_n}-1)\alpha}{q_n([n]_{q_n}-1)([n]_{q_n}+\beta)^2}\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{\alpha^2}{([n]_{q_n}+\beta)^2}$$

$$\leq\frac{(1-q_n^n)[n]_{q_n}^2-q_n(2\beta-1)[n]_{q_n}^2-q_n\beta(\beta-1)[n]_{q_n}+q_n\beta^2}{q_n([n]_{q_n}-1)([n]_{q_n}+\beta)^2}$$

$$+\left(\frac{[n]_{q_n}^2+2q_n[n]_{q_n}([n]_{q_n}-1)\alpha}{q_n([n]_{q_n}-1)([n]_{q_n}+\beta)^2}\right)+\frac{\alpha^2}{([n]_{q_n}+\beta)^2},$$

which implies that

$$\lim_{n\to\infty}\|L_{n,q_n}^{(\alpha,\beta)}(e_2;x)-e_2\|_{x^2}=0.$$

Thus the proof is completed. □

We give the following theorem to approximate all functions in $C_{x^2}[0,\infty)$. These type of results are given in [33] for locally integrable functions.

Theorem 4.9. *Let $q=q_n$ satisfy $0<q_n<1$ and let $q_n\to 1$ as $n\to\infty$. For each $f\in C_{x^2}[0,\infty)$ and $\alpha>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|L_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)\right|}{(1+x^2)^{1+\alpha^2}}=0.$$

Proof. For any fixed $x_0>0$,

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{\left|L_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)\right|}{(1+x^2)^{1+\alpha^2}}&=\sup_{x\leq x_0}\frac{\left|L_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)\right|}{(1+x^2)^{1+\alpha^2}}+\sup_{x\geq x_0}\frac{\left|L_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)\right|}{(1+x^2)^{1+\alpha^2}}\\&\leq\|L_{n,q_n}^{(\alpha,\beta)}(f)-f\|_{C[0,a]}+\|f\|_{x^2}\sup_{x\geq x_0}\frac{\left|L_{n,q_n}^{(\alpha,\beta)}(1+t^2;x)\right|}{(1+x^2)^{1+\alpha^2}}+\sup_{x\geq x_0}\frac{|f(x)|}{(1+x^2)^{1+\alpha^2}}.\end{aligned}$$

It follows from Theorem 4.7 that the first term of the above inequality tends to zero. By Lemma 4.4, for any fixed $x_0>0$ it is easily seen that $\sup\limits_{x\geq x_0}\frac{|L_{n,q_n}^{(\alpha,\beta)}(1+t^2;x)|}{(1+x^2)^{1+\alpha^2}}$ tends to zero as $n\to\infty$. We can choose $x_0>0$ to be sufficiently large so that the last part of the above inequality can be made arbitrarily small and this proves the theorem. □

4.2.2 Pointwise Estimates

Now, we establish some pointwise estimates of the rate of convergence of the q-Stancu-Beta operators. First, we give the relationship between the local smoothness of f and local approximation.

We know that a function $f\in C[0,\infty)$ is $Lip_{M_f}(\alpha)$ on E, $\alpha\in(0,1]$, $E\subset[0,\infty)$ if it satisfies the condition

$$|f(t)-f(x)|\leq M_f|t-x|^\alpha,\quad t\in[0,\infty)\text{ and }x\in E, \tag{4.7}$$

where M_f is a constant depending only on α and f.

Theorem 4.10. *Let $f \in Lip(\alpha)$, $E \subset [0,\infty)$ and $\alpha \in (0,1]$. We have*

$$\begin{aligned}
&|L_{n,q}^{(\alpha,\beta)}(f;x) - f(x)| \\
&\leq M_f\left\{\left[\left(\frac{[n]_q}{q([n]_q-1)} - \frac{[n]_q-\beta}{[n]_q+\beta}\right)x^2 + \frac{x}{q([n]_q-1)} + \frac{\alpha^2}{([n]_q+\beta^2)}\right]^{\frac{\alpha}{2}} + 2\big(d(x,E)\big)^{\alpha}\right\},
\end{aligned}$$

$x \in [0,\infty)$ *where $d(x,E)$ represents the distance between x and E defined as*

$$d(x,E) = \inf\{|x-y| : y \in E\}.$$

Proof. For $x_0 \in \overline{E}$, the closure of the set $E \in [0,\infty)$, we have

$$|f(t) - f(x)| \leq |f(t) - f(x_0)| + |f(x_0) - f(x)|, \quad x \in [0,\infty).$$

By (4.7) we get

$$\begin{aligned}
&|L_{n,q}^{(\alpha,\beta)}(f;x) - f(x)| \\
&= |L_{n,q}^{(\alpha,\beta)}(f;x) - L_{n,q}^{(\alpha,\beta)}\big(f(x);x\big)| \leq L_{n,q}^{(\alpha,\beta)}\big(|f(t)-f(x)|;x\big) \\
&\leq L_{n,q}^{(\alpha,\beta)}\big(|f(t)-f(x_0)|;x\big) + L_{n,q}^{(\alpha,\beta)}\big(|f(x)-f(x_0)|;x\big) \\
&\leq L_{n,q}^{(\alpha,\beta)}\big(|f(t)-f(x_0)|;x\big) + |f(x)-f(x_0)| \\
&\leq M_f\big\{L_{n,q}^{(\alpha,\beta)}\big(|t-x_0|^{\alpha};x\big) + |x-x_0|^{\alpha}\big\} \\
&\leq M_f\big\{L_{n,q}^{(\alpha,\beta)}\big(|t-x|^{\alpha} + |x-x_0|^{\alpha};x\big) + |x-x_0|^{\alpha}\big\} \\
&= M_f\big\{L_{n,q}^{(\alpha,\beta)}\big(|t-x|^{\alpha};x\big) + 2|x-x_0|^{\alpha}\big\}.
\end{aligned}$$

Using the Hölder's inequality and Lemma 4.6, we find that

$$\begin{aligned}
&|L_{n,q}^{(\alpha,\beta)}(f;x) - f(x)| \\
&\leq M_f\left\{\left(L_{n,q}^{(\alpha,\beta)}\big(|t-x|^{\alpha p};x\big)\right)^{\frac{1}{p}}\left(L_{n,q}^{(\alpha,\beta)}\big(1^q;x\big)\right)^{\frac{1}{q}} + 2\big(d(x,E)\big)^{\alpha}\right\} \\
&= M_f\left\{\left(L_{n,q}^{(\alpha,\beta)}\big(|t-x|^{2};x\big)\right)^{\frac{\alpha}{2}} + 2\big(d(x,E)\big)^{\alpha}\right\} \\
&= M_f\left\{\left[\left(\frac{[n]_q}{q([n]_q-1)} - \frac{[n]_q-\beta}{[n]_q+\beta}\right)x^2 + \frac{x}{q([n]_q-1)} + \frac{\alpha^2}{([n]_q+\beta^2)}\right]^{\frac{\alpha}{2}} + 2\big(d(x,E)\big)^{\alpha}\right\} \\
&= M_f\left\{\delta_n(x)^{\frac{\alpha}{2}} + 2\big(d(x,E)\big)^{\alpha}\right\}.
\end{aligned}$$

□

Now, we give local direct estimate for the q-Stancu-Beta operators using the Lipschitz-type maximal function of order α introduced by B. Lenze [63] as

$$\tilde{\omega}_\alpha(f,x) = \sup_{t\neq x,\ t\in[0,\infty)} \frac{|f(t)-f(x)|}{|t-x|^\alpha}, \quad x\in[0,\infty) \text{ and } \alpha\in(0,1]. \tag{4.8}$$

Theorem 4.11. *Let $\alpha\in(0,1]$ and $f\in C_B[0,\infty)$. Then for all $x\in[0,\infty)$, we have*

$$\begin{aligned}&|L_{n,q}^{(\alpha,\beta)}(f;x)-f(x)|\\ &\le \tilde{\omega}_\alpha(f,x)\left\{\left(\frac{[n]_q}{q([n]_q-1)}-\frac{[n]_q-\beta}{[n]_q+\beta}\right)x^2+\frac{x}{q([n]_q-1)}+\frac{\alpha^2}{([n]_q+\beta^2)}\right\}^{\frac{\alpha}{2}}.\end{aligned}$$

Proof. From (4.8) we have

$$|f(t)-f(x)|\le\tilde{\omega}_\alpha(f,x)|t-x|^\alpha$$

and

$$\begin{aligned}&|L_{n,q}^{(\alpha,\beta)}(f;x)-f(x)|\\ &\le L_{n,q}^{(\alpha,\beta)}\big(|f(t)-f(x)|;x\big)\le\tilde{\omega}_\alpha(f,x)L_{n,q}^{(\alpha,\beta)}\big(|t-x|^\alpha;x\big).\end{aligned}$$

Applying Hölder's inequality, we have

$$|L_{n,q}^{(\alpha,\beta)}(f;x)-f(x)|\le\tilde{\omega}_\alpha(f,x)L_{n,q}^{(\alpha,\beta)}\big(|t-x|^2;x\big)^{\frac{\alpha}{2}}.$$

Using Lemma 4.6, we then have our assertion. □

4.2.3 Voronovskaja type theorem

In this section we prove Voronovskaja type results for q-Stancu-Beta type operators.

Lemma 4.12. *Assume that $q_n\in(0,1)$ and $q_n^n\to a$, $(0\le a<1)$ as $n\to\infty$. For every $x\in[0,\infty)$ there hold*

$$\lim_{n\to\infty}[n]_{q_n}L_{n,q_n}^{(\alpha,\beta)}(e_1-e_0x;x)=\alpha-\beta x,$$

$$\lim_{n\to\infty}[n]_{q_n}L_{n,q_n}^{(\alpha,\beta)}\big((e_1-e_0x)^2;x\big)=(2-a)x^2+x.$$

Theorem 4.13. *Assume that $q_n\in(0,1)$, $q_n\to1$ and $q_n^n\to a$ $(0\le a<1)$ as $n\to\infty$. For any $f\in C_2^*[0,\infty)$ such that $f',f''\in C_2^*[0,\infty)$ the following equality holds*

$$\lim_{n\to\infty}[n]_{q_n}\left(L_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)\right)=(\alpha-\beta x)f'(x)+\frac{(2-a)x^2+x}{2}f''(x)$$

uniformly on any $[0,A]$, $A>0$.

Proof. Let $f, f', f'' \in C_2^*[0,\infty)$ and $x \in [0,\infty)$ be fixed. By the Taylor's formula we may write

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + r(t;x)(t-x)^2, \tag{4.9}$$

where $r(t;x)$ is the Peano form of remainder, $r(.;x) \in C_2^*[0,\infty)$ and $\lim_{t\to x} r(t;x) = 0$. Applying $L_{n,q_n}^{(\alpha,\beta)}$ to (4.9) we obtain

$$\begin{aligned}
&[n]_{q_n}\left(L_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right)\\
&= [n]_{q_n} L_{n,q_n}^{(\alpha,\beta)}(e_1 - e_0 x; x) f'(x)\\
&+ \frac{1}{2}[n]_{q_n} L_{n,q_n}^{(\alpha,\beta)}\left((e_1 - e_0 x)^2; x\right) f''(x) + [n]_{q_n} L_{n,q_n}^{(\alpha,\beta)}\left(r(.;x)(.-x)^2; x\right).
\end{aligned}$$

By the Cauchy-Schwartz inequality, we have

$$L_{n,q_n}^{(\alpha,\beta)}\left(r(.;x)(.-x)^2;x\right) \le \sqrt{L_{n,q_n}^{(\alpha,\beta)}\left(r^2(.;x);x\right)}\sqrt{L_{n,q_n}^{(\alpha,\beta)}\left((.-x)^4;x\right)}. \tag{4.10}$$

Observe that $r^2(x;x) = 0$ and $r^2(.;x) \in C_2^*[0,\infty)$. Then it follows from Theorem 4.8 that

$$\lim_{n\to\infty} L_{n,q_n}^{(\alpha,\beta)}\left(r^2(.;x);x\right) = r^2(x;x) = 0 \tag{4.11}$$

uniformly with respect to $x \in [0, A]$. Now from (4.10), (4.11) and Lemma 4.12 above we immediately get

$$\lim_{n\to\infty} [n]_{q_n} L_{n,q_n}^{(\alpha,\beta)}\left(r(.;x)(.-x)^2;x\right) = 0.$$

Thus we get the following

$$\begin{aligned}
&\lim_{n\to\infty} [n]_{q_n}\left(L_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right)\\
&= \lim_{n\to\infty}\Big(f'(x)[n]_{q_n} L_{n,q_n}^{(\alpha,\beta)}(e_1 - e_0 x; x) + \frac{1}{2} f''(x)[n]_{q_n} L_{n,q_n}^{(\alpha,\beta)}\left((e_1 - e_0 x)^2; x\right)\\
&+ [n]_{q_n} L_{n,q_n}^{(\alpha,\beta)}\left(r(.;x)(.-x)^2;x\right)\Big)\\
&= (\alpha - \beta x) f'(x) + \frac{(2-a)x^2 + x}{2} f''(x).
\end{aligned}$$

This completes the proof. □

4.3 q-Stancu-Beta operators preserving x^2

It has been observed that most of the approximating operators, L_n, preserve $e_i(x) = x^i$ $(i = 0, 1)$, i.e., $L_n(e_0;x) = e_0(x)$ and $L_n(e_1;x) = e_1(x)$, $n \in \mathbb{N}$. These conditions hold specially, for the Bernstein polynomials, Szász-Mirakjan operators, and the Baskakov operators (see [2],[61]). For each of these operators $L_n(e_2;x) \neq e_2(x) = x^2$. King [58] has presented a non-trivial sequence $\{V_n\}$ of positive linear operators which approximate each continuous function on $[0,1]$ while preserving the functions e_0 and e_2, i.e., $V_n : C[0,1] \to C[0,1]$, for any $n \in \mathbb{N}$ and for any function $f \in C[0,1]$, is given as follows

$$V_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}(r_n^*(x))^k\,(1-r_n^*(x))^{n-k},\ \ 0 \le x \le 1, \tag{4.12}$$

where $r_n^*(x) : [0,1] \to [0,1]$ are defined by

$$r_n^*(x)\begin{cases} x^2, & n=1; \\ -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}, & n = 2,3,\ldots. \end{cases} \tag{4.13}$$

These are reduced to the Bernstein polynomials if we replace $r_n^*(x)$ by e_1. This sequence satisfies $V_n(e_1;x) = r_n^*(x)$ and preserves the test functions e_0, e_2. It is observed that the operators V_n have a better rate of convergence than the classical Bernstein polynomials whenever $0 \le x \le 1/3$. Operators which preserve x^2 have been investigated by several authors, e.g. [28] and [71]. Recently Cai [17] introduced a new kind of modification of q-Stancu-Beta operators which preserve x^2 based on the concept of q-integers and proved some approximation properties for these operators.

Cai [17] transformed the operators (4.1) in order to preserve the quadratic function e_2 by defining the functions

$$v_{n,q}(x) = \frac{-1+\sqrt{4q[n]_q([n]_q-1)x^2+1}}{2[n]_q},\ \ x \ge 0. \tag{4.14}$$

He considered the following positive linear operators.

Definition 4.14. *Let $q \in (0,1)$ and $n \in \mathbb{N}$. For $f : [0,\infty) \to \mathbb{R}$, the q-Stancu-Beta operators are defined as*

$$\begin{aligned} L_{n,q}^*(f;x) &:= L_n^q(f;v_{n,q}(x)) \\ &= \frac{K(A,[n]_q v_{n,q}(x))}{B_q([n]_q v_{n,q}(x),[n]_q+1)}\int_0^{\infty/A}\frac{u^{[n]_q v_{n,q}(x)-1}}{(1+u)_q^{[n]_q v_{n,q}(x)+[n]_q+1}}f\left(q^{[n]_q v_{n,q}(x)}u\right)d_qu. \end{aligned} \tag{4.15}$$

Moments $L_{n,q}^*(e_m;x)$ are of particular importance in approximation theory by positive operators.

Lemma 4.15. *([17]) The operators defined by (4.15) verify the following identities*

$$L_{n,q}^*(e_0;x)=1,\quad L_{n,q}^*(e_1;x)=v_{n,q}(x),\quad L_{n,q}^*(e_2;x)=x^2,$$

$$L_{n,q}^*\left((e_1-e_0x)^2;x\right)=2x\left(x-v_{n,q}(x)\right).$$

Further, he obtained a local approximation theorem, and got the pointwise convergence rate theorem and also a weighted approximation theorem etc.

Now we come to our results in the continuation of [17]. First we determine $L_{n,q}^*(e_3;x)$ and $L_{n,q}^*(e_4;x)$.

Lemma 4.16. *We have*

$$L_{n,q}^*(e_3;x)=\frac{1}{q^3([n]_q-1)([n]_q-2)}\left([n]_q^2v_{n,q}^3(x)+3[n]_qv_{n,q}^2(x)+2v_{n,q}(x)\right),$$

$$\begin{aligned}L_{n,q}^*(e_4;x) &= \frac{1}{q^6([n]_q-1)([n]_q-2)([n]_q-3)}\\ &\quad\times\left([n_q]^3v_{n,q}^4(x)+6[n]_q^2v_{n,q}^3(x)+11[n]_qv_{n,q}^2(x)+6v_{n,q}(x)\right).\end{aligned}$$

Proof. By the definition of modified q-Stancu-Beta operators, we have

$$\begin{aligned}L_{n,q}^*(e_3;x) &= \frac{K\left(A,[n]_qv_{n,q}(x)\right)q^{3[n]_qv_{n,q}(x)}}{B_q\left([n]_qv_{n,q}(x),[n]_q+1\right)}\int_0^{\infty/A}\frac{u^{[n]_qv_{n,q}(x)+2}}{(1+u)_q^{[n]_qv_{n,q}(x)+[n]_q+1}}\,\mathrm{d}_qu\\
&= \frac{K\left(A,[n]_qv_{n,q}(x)\right)q^{3[n]_qv_{n,q}(x)}}{B_q\left([n]_qv_{n,q}(x),[n]_q+1\right)}\frac{B_q\left([n]_qv_{n,q}(x)+3,[n]_q-2\right)}{K\left(A,[n]_qv_{n,q}(x)+3\right)}\\
&= q^{3[n]_qv_{n,q}(x)}K\left(A,[n]_qv_{n,q}(x)\right)\frac{\Gamma_q\left([n]_qv_{n,q}(x)+[n]_q+1\right)}{\Gamma_q\left([n]_qv_{n,q}(x)\right)\Gamma_q\left([n]_q+1\right)}\\
&\quad\times\frac{1}{q^{3[n]_qv_{n,q}(x)+3}K\left(A,[n]_qv_{n,q}(x)\right)}\frac{\Gamma_q\left([n]_qv_{n,q}(x)+3\right)\Gamma_q\left([n]_q-2\right)}{\Gamma_q\left([n]_qv_{n,q}(x)+[n]_q+1\right)}\\
&= \frac{\Gamma_q\left([n]_qv_{n,q}(x)+3\right)\Gamma_q\left([n]_q-2\right)}{\Gamma_q\left([n]_qv_{n,q}(x)\right)\Gamma_q\left([n]_q+1\right)}\frac{1}{q^3}\\
&= \frac{\left([n]_qv_{n,q}(x)+2\right)\left([n]_qv_{n,q}(x)+1\right)[n]_qv_{n,q}(x)\Gamma_q\left([n]_qv_{n,q}(x)\right)\Gamma_q\left([n]_q-2\right)}{\left([n]_qv_{n,q}(x)\right)[n]_q\left([n]_q-1\right)\left([n]_q-2\right)\Gamma_q\left([n]_q-2\right)}\frac{1}{q^3}\\
&= \frac{v_{n,q}(x)\left([n]_qv_{n,q}(x)+1\right)\left([n]_qv_{n,q}(x)+2\right)}{q^3\left([n]_q-1\right)\left([n]_q-2\right)}\\
&= \frac{1}{q^3\left([n]_q-1\right)\left([n]_q-2\right)}\left([n]_q^2v_{n,q}^3(x)+3[n]_qv_{n,q}^2(x)+2v_{n,q}(x)\right),\end{aligned}$$

and

$$
\begin{aligned}
L^*_{n,q}(e_4;x) &= \frac{K\big(A,[n]_q v_{n,q}(x)\big)q^{4[n]_q v_{n,q}(x)}}{B_q\big([n]_q v_{n,q}(x),[n]_q+1\big)}\int_0^{\infty/A}\frac{u^{[n]_q v_{n,q}(x)+3}}{(1+u)_q^{[n]_q v_{n,q}(x)+[n]_q+1}}\mathrm{d}_q u\\
&= \frac{K\big(A,[n]_q v_{n,q}(x)\big)q^{4[n]_q v_{n,q}(x)}}{B_q\big([n]_q v_{n,q}(x),[n]_q+1\big)}\frac{B_q\big([n]_q v_{n,q}(x)+4,[n]_q-3\big)}{K\big(A,[n]_q v_{n,q}(x)+4\big)}\\
&= q^{4[n]_q v_{n,q}(x)}K\big(A,[n]_q v_{n,q}(x)\big)\frac{\Gamma_q\big([n]_q v_{n,q}(x)+[n]_q+1\big)}{\Gamma_q\big([n]_q v_{n,q}(x)\big)\Gamma_q\big([n]_q+1\big)}\\
&\times\frac{1}{q^{4[n]_q v_{n,q}(x)+6}K\big(A,[n]_q v_{n,q}(x)\big)}\frac{\Gamma_q\big([n]_q v_{n,q}(x)+4\big)\Gamma_q\big([n]_q-3\big)}{\Gamma_q\big([n]_q v_{n,q}(x)+[n]_q+1\big)}\\
&= \frac{\Gamma_q\big([n]_q v_{n,q}(x)+4\big)\Gamma_q\big([n]_q-3\big)}{\Gamma_q\big([n]_q v_{n,q}(x)\big)\Gamma_q\big([n]_q+1\big)}\frac{1}{q^6}\\
&= \frac{\big([n]_q v_{n,q}(x)+3\big)\big([n]_q v_{n,q}(x)+2\big)\big([n]_q v_{n,q}(x)+1\big)[n]_q v_{n,q}(x)\Gamma_q\big([n]_q v_{n,q}(x)\big)\Gamma_q\big([n]_q-3\big)}{q^6\big([n]_q v_{n,q}(x)\big)[n]_q\big([n]_q-1\big)\big([n]_q-3\big)\Gamma_q\big([n]_q-3\big)}\\
&= \frac{v_{n,q}(x)\big([n]_q v_{n,q}(x)+1\big)\big([n]_q v_{n,q}(x)+2\big)\big([n]_q v_{n,q}(x)+3\big)}{q^6\big([n]_q-1\big)\big([n]_q-2\big)\big([n]_q-3\big)}\\
&= \frac{1}{q^6\big([n]_q-1\big)\big([n]_q-2\big)\big([n]_q-3\big)}\\
&\times\big([n]_q^3v_{n,q}^4(x)+6[n]_q^2v_{n,q}^3(x)+11[n]_q v_{n,q}^2(x)+6v_{n,q}(x)\big).
\end{aligned}
$$

□

Remark 4.17. *Suppose that $q\in(0,1)$; then for $x\in[0,\infty)$, we can have the following formula for the m^{th} order moment from the operators defined by (4.15):*

$$L^*_{n,q}(e_m;x)=\frac{\Gamma_q\big([n]_q v_{n,q}(x)+m\big)\Gamma_q\big([n]_q-m+1\big)}{\Gamma_q\big([n]_q v_{n,q}(x)\big)\Gamma_q\big([n]_q+1\big)q^{m(m-1)/2}}.$$

Lemma 4.18. *Let $v_{n,q}$, $n\in\mathbb{N}$ be defined by (4.14), where $q\in(0,1)$. Then the following statements hold.*

(i) $v_{n,q}(0)=0$;

(ii) $0\le v_{n,q}(x)\le x$;

(iii) $x-v_{n,q}(x)$ *is strictly increasing in x and*

$$x-v_{n,q}(x)\le\frac{1}{[n]_q}\frac{(1+q-q^n)x+1}{(1+\sqrt{q})},$$

$$L^*_{n,q}\left((e_1-e_0x)^2;x\right)\le\frac{2}{[n]_q}\frac{(1+q-q^n)x^2+x}{(1+\sqrt{q})}.$$

Proof. For proving (iii) we can consider the function $h : [0,\infty) \to \mathbb{R},\ h(x) = x - v_{n,q}(x)$. It is clear that h is strictly increasing and

$$\begin{aligned}
0 \le h(x) &= \frac{1}{2[n]_q}\left(2[n]_q x + 1 - \sqrt{4q[n]_q([n]_q - 1)x^2 + 1}\right) \\
&= \frac{1}{2[n]_q}\frac{4(1-q^n)[n]_q x^2 + 4q[n]_q x^2 + 4[n]_q x}{2[n]_q x + 1 + \sqrt{4q[n]_q([n]_q - 1)x^2 + 1}} \\
&\le \frac{1}{2[n]_q}\frac{4(1+q-q^n)[n]_q x^2 + 4[n]_q x}{2[n]_q x + 2\sqrt{q}([n]_q - 1)x} \\
&\le \frac{1}{[n]_q}\frac{(1+q-q^n)x + 1}{(1+\sqrt{q})}.
\end{aligned}$$

□

Lemma 4.19. *Assume that $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a$ $(0 \le a < 1)$ as $n \to \infty$. For every $x \in [0,\infty)$, we have*

$$\lim_{n\to\infty} [n]_{q_n} L^*_{n,q_n}(e_1 - e_0 x; x) = -\frac{(2-a)x+1}{2}, \tag{4.16}$$

$$\lim_{n\to\infty} [n]_{q_n} L^*_{n,q_n}\left((e_1 - e_0 x)^2; x\right) = (2-a)x^2 + x, \tag{4.17}$$

$$\lim_{n\to\infty} [n]^2_{q_n} L^*_{n,q_n}\left((e_1 - e_0 x)^4; x\right) = 3x^2 + 6(2-a)x^3 + 3(2-a)^2 x^4. \tag{4.18}$$

Proof. The proof is based on the following limit

$$\lim_{n\to\infty} [n]_{q_n}\left(x - v_{n,q_n}(x)\right) = -\frac{(2-a)x+1}{2}.$$

We prove (4.18). To do this we give an explicit formula for $L^*_{n,q_n}\left((e_1 - e_0 x)^4; x\right)$. With the help of Lemma 4.15 and Lemma 4.16, we have

$$\begin{aligned}
&L^*_{n,q_n}\left((e_1 - e_0 x)^4; x\right) \\
&= L^*_{n,q_n}(e_4; x) - 4x L^*_{n,q_n}(e_3; x) + 6x^2 L^*_{n,q_n}(e_2; x) - 4x^3 L^*_{n,q_n}(e_1; x) + x^4 \\
&= \frac{[n]^3_{q_n} v^4_{n,q_n}(x) + 6[n]^2_{q_n} v^3_{n,q_n}(x) + 11[n]_{q_n} v^2_{n,q_n}(x) + 6v_{n,q_n}(x)}{q_n^6([n]_{q_n} - 1)([n]_{q_n} - 2)([n]_{q_n} - 3)} \\
&\quad - 4x\left(\frac{[n]^2_{q_n} v^3_{n,q_n}(x) + 3[n]_{q_n} v^2_{n,q_n}(x) + 2v_{n,q_n}(x)}{q_n^3([n]_{q_n} - 1)([n]_{q_n} - 2)}\right) \\
&\quad + 6x^2\left(\frac{[n]_{q_n} v^2_{n,q_n}(x) + v_{n,q_n}(x)}{q_n([n]_{q_n} - 1)}\right) - 4x^3 v_{n,q_n}(x) + x^4 \\
&= \frac{6v_{n,q_n}(x)}{q_n^6([n]_{q_n} - 1)([n]_{q_n} - 2)([n]_{q_n} - 3)}
\end{aligned}$$

$$
\begin{aligned}
&+\left\{\frac{11[n]_{q_n}v_{n,q_n}^2(x)}{q_n^6([n]_{q_n}-1)([n]_{q_n}-2)([n]_{q_n}-3)}-\frac{8xv_{n,q_n}(x)}{q_n^3([n]_{q_n}-1)([n]_{q_n}-2)}\right\}\\
&+\left\{\frac{6[n]_{q_n}^2v_{n,q_n}^3(x)}{q_n^6([n]_{q_n}-1)([n]_{q_n}-2)([n]_{q_n}-3)}-\frac{12[n]_{q_n}xv_{n,q_n}^2(x)}{q_n^3([n]_{q_n}-1)([n]_{q_n}-2)}+\frac{6x^2v_{n,q_n}(x)}{q_n([n]_{q_n}-1)}\right\}\\
&+\left\{\begin{array}{c}\frac{[n]_{q_n}^3v_{n,q_n}^4(x)}{q_n^6([n]_{q_n}-1)([n]_{q_n}-2)([n]_{q_n}-3)}-\frac{4[n]_{q_n}^2xv_{n,q_n}^3(x)}{q_n^3([n]_{q_n}-1)([n]_{q_n}-2)}+\frac{6[n]_{q_n}x^2v_{n,q_n}^2(x)}{q_n([n]_{q_n}-1)}\\ -4x^3v_{n,q_n}(x)+x^4\end{array}\right\}\\
&:=A_{n,q_n}(x)+B_{n,q_n}(x)+C_{n,q_n}(x)+D_{n,q_n}(x).
\end{aligned}
$$

Simple but tedious calculations show that

$$
\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n}^2L_{n,q_n}^*\left((e_1-e_0x)^4;x\right)\\
&=\lim_{n\to\infty}[n]_{q_n}^2\left(A_{n,q_n}(x)+B_{n,q_n}(x)+C_{n,q_n}(x)+D_{n,q_n}(x)\right)\\
&=0+3x^2+6(2-a)x^3+3(2-a)^2x^4.
\end{aligned}
$$

Let us show the details of the third limit.

$$
\begin{aligned}
&[n]_{q_n}^2C_{n,q_n}(x)=\frac{6[n]_{q_n}^4(v_{n,q_n}(x)-x)v_{n,q_n}^2(x)}{q_n^6([n]_{q_n}-1)([n]_{q_n}-2)([n]_{q_n}-3)}\\
&+\left(\frac{6[n]_{q_n}^4}{q_n^6([n]_{q_n}-1)([n]_{q_n}-2)([n]_{q_n}-3)}-\frac{12[n]_{q_n}^3}{q_n^3([n]_{q_n}-1)([n]_{q_n}-2)}\right)(v_{n,q_n}(x)-x)v_{n,q_n}(x)\\
&+\left(\frac{6[n]_{q_n}^4}{q_n^6([n]_{q_n}-1)([n]_{q_n}-2)([n]_{q_n}-3)}-\frac{12[n]_{q_n}^3}{q_n^3([n]_{q_n}-1)([n]_{q_n}-2)}-\frac{6[n]_{q_n}^2}{q_n([n]_{q_n}-1)}\right)xv_{n,q_n}(x)\\
&\to-6\frac{(2-a)x+1}{2}x^2-6\frac{(2-a)x+1}{2}x^2+12\frac{(2-a)x+1}{2}x^2+6(2-a)x^3\\
&=6(2-a)x^3.
\end{aligned}
$$

□

Lemma 4.20. *Let $m\in\mathbb{N}\cup\{0\}$ and $q\in(0,1)$ be fixed. Then there exists a positive constant $K_m(q)$ such that*

$$\|L_{n,q}^*(1+e_m;x)\|_m\le K_m(q),\ \ n\in\mathbb{N}. \tag{4.19}$$

Moreover, for every $f\in C_m^*[0,\infty)$, we have

$$\|L_{n,q}^*(f)\|_m\le K_m(q)\|f\|_m,\ \ n\in\mathbb{N}. \tag{4.20}$$

Thus $L_{n,q}^*$ is a positive linear operator from $C_m^*[0,\infty)$ into $C_m^*[0,\infty)$ for any $m\in\mathbb{N}\cup\{0\}$.

Proof. The inequality (4.19) is obvious for $m = 0$. Let $m \geq 1$. Then by Remark 4.17, we have

$$\begin{aligned}
&\frac{1}{(1+x^m)} L^*_{n,q}(1+e_m;x) \\
&= \frac{1}{(1+x^m)} + \frac{1}{(1+x^m)} \frac{\Gamma_q\big([n]_q v_{n,q}(x)+m\big)\Gamma_q\big([n]_q - m+1\big)}{\Gamma_q\big([n]_q v_{n,q}(x)\big)\Gamma_q\big([n]_q+1\big) q^{m(m-1)/2}} \\
&\leq 1 + c_m(q) = K_m(q),
\end{aligned}$$

where $c_m(q)$ and $K_m(q)$ are positive constants depending on m and q. Hence (4.19) follows. On the other hand, we have

$$\|L^*_{n,q}(f)\|_m \leq \|f\|_m \|L^*_{n,q}(1+e_m)\|_m$$

for every $f \in C^*_m[0,\infty)$. By applying (4.19), we obtain (4.20). $\square$

4.3.1 Convergence of modified operators

In this section we study approximation properties of modified q-Stancu-Beta operators.

Theorem 4.21. *Let $q_n \in (0,1)$. Then the sequence $\{L^*_{n,q_n}(f)\}$ converges to f uniformly on $[0,A]$ for each $f \in C^*_2[0,\infty)$ if and only if $\lim_{n\to\infty} q_n = 1$.*

Proof. Assume that $\lim_{n\to\infty} q_n = 1$. Fix $A > 0$ and consider the lattice homomorphism $T_A : C[0,\infty) \to C[0,A]$ defined by

$$T_A(f) := f_{|[0,A]}.$$

We see that

$$T_A\left(L^*_{n,q_n}(e_0)\right) = T_A(1), \quad T_A\left(L^*_{n,q_n}(e_1)\right) = T_A(t), \quad T_A\left(L^*_{n,q_n}(e_2)\right) = T_A(t^2)$$

uniformly on $[0,A]$. From Proposition 4.2.5, (6) of [2], we have that $C^*_2[0,\infty)$ is isomorphic to $C[0,1]$ and the set $\{1,t,t^2\}$ is a Korovkin set in $C^*_2[0,\infty)$. So the universal Korovkin type property ([2] property (vi) of Theorem 4.1.4) implies that

$$L^*_{n,q_n}(f;x) \to f(x) \quad \text{uniformly on } [0,A] \text{ as } n \to \infty$$

provided $f \in C^*_2[0,\infty)$ and $A > 0$.

We prove the converse result by contradiction. Assume that $\{q_n\}$ does not converge to 1. Then it has to contain a subsequence $\{q_{n_k}\} \subset (0,1)$ such that $q_{n_k} \to \alpha \in [0,1)$ as $k \to \infty$. So

$$\frac{1}{[n_k]_{q_{n_k}}} = \frac{1-q_{n_k}}{1-(q_{n_k})^{n_k}} \to 1-\alpha \quad \text{as } k \to \infty$$

and we get

$$\begin{aligned} L^*_{n,q_{n_k}}(e_1;x) - x &= v_{n,q_{n_k}}(x) - x \\ &= \frac{-1+\sqrt{4q_{n_k}[n]_{q_{n_k}}\left([n]_{q_{n_k}}-1\right)x^2+1}}{2[n]_{q_{n_k}}} - x \\ &\to \frac{-(1-\alpha)+\sqrt{4\alpha^2x^2+(1-\alpha)^2}}{2(1-\alpha)} - x \neq 0. \end{aligned}$$

This leads to a contradiction. Thus $q_n \to 1$ as $n \to \infty$. Hence theorem is proved. □

4.3.2 Rate of global convergence

Next, we obtain a direct approximation theorem in $C^*_m[0,\infty)$ and an estimation in terms of the weighted modulus of continuity. It is known that, if f is not uniformly continuous on the interval $[0,\infty)$, then the usual first modulus of continuity $\omega(f,\delta)$ does not tend to zero, as $\delta \to 0$. For every $f \in C^*_m[0,\infty)$ the weighted modulus of continuity is defined as follows

$$\Omega_m(f,\delta) = \sup_{x\geq 0,\ 0<h\leq\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^m}.$$

Lemma 4.22. *([64]) Let $f \in C^*_m[0,\infty)$, $m \in \mathbb{N}$. Then*

(1) $\Omega_m(f,\delta)$ is a monotone increasing function of δ,

(2) $\lim_{\delta\to 0^+}\Omega_m(f,\delta) = 0$,

(3) for any $\alpha \in [0,\infty)$, $\Omega_m(f,\alpha\delta) \leq (1+\alpha)\Omega_m(f,\delta)$.

The following is the main convergence result of this chapter in which we give an expression of the approximation error with the operators $L^*_{n,q}$ by means of Ω_m.

Theorem 4.23. *If $f \in C^*_m[0,\infty)$ then*

$$\|L^*_{n,q}(f) - f\|_{m+1} \leq k\Omega_m\left(f; \sqrt{\frac{2(1+q)}{(1+\sqrt{q})[n]_q}}\right),$$

where k is a constant independent of f and n.

Proof. From the definition of $\Omega_m(f,\delta)$ and Lemma 4.22, we may write

$$\begin{aligned}
&|f(t)-f(x)| \\
&\le \left(1+(x+|t-x|)^m\right)\left(\frac{|t-x|}{\delta}+1\right)\Omega_m(f,\delta) \\
&\le \left(1+(2x+t)^m\right)\left(\frac{|t-x|}{\delta}+1\right)\Omega_m(f,\delta) \\
&:= \mu_x(t)\left(\frac{|t-x|}{\delta}+1\right)\Omega_m(f,\delta).
\end{aligned}$$

Then

$$|L^*_{n,q}(f;x)-f(x)| \le \Omega_m(f,\delta)\left(L^*_{n,q}(\mu_x(t);x)+L^*_{n,q}\left(\mu_x(t)\frac{|t-x|}{\delta};x\right)\right).$$

Applying the Cauchy-Schwartz inequality to the second term, we get

$$L^*_{n,q}\left(\mu_x(t)\frac{|t-x|}{\delta};x\right) \le \left(L^*_{n,q}(\mu_x^2(t);x)\right)^{1/2}\left(L^*_{n,q}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2}.$$

Consequently

$$\begin{aligned}
&|L^*_{n,q}(f;x)-f(x)| \\
&\le \left(L^*_{n,q}(\mu_x(t);x)+\left(L^*_{n,q}(\mu_x^2(t);x)\right)^{1/2}\left(L^*_{n,q}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2}\right)\Omega_m(f,\delta). \qquad (4.21)
\end{aligned}$$

Notice that by Lemma 4.20 there are positive constants $K_m(q)$ and $K^1_m(q)$ such that

$$L^*_{n,q}(\mu_x(t);x) \le K_m(q)(1+x^m), \qquad \left(L^*_{n,q}(\mu_x^2(t);x)\right)^{1/2} \le K^1_m(q)(1+x^m). \qquad (4.22)$$

Hence we get

$$\begin{aligned}
\left(L^*_{n,q}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2} &\le \frac{1}{\delta}\sqrt{\frac{2}{(1+\sqrt{q})[n]_q}\left((1+q-q^n)x^2+x\right)} \\
&\le \frac{1}{\delta}\sqrt{\frac{2}{(1+\sqrt{q})[n]_q}\left((1+q)x^2+x\right)} \\
&\le \frac{1}{\delta}\sqrt{\frac{2(1+q)}{(1+\sqrt{q})[n]_q}}\sqrt{x^2+x} \\
&\le \frac{1}{\delta}\sqrt{\frac{2(1+q)}{(1+\sqrt{q})[n]_q}}(1+x). \qquad (4.23)
\end{aligned}$$

Now from (4.21)-(4.23) we have

$$
\begin{aligned}
&|L^*_{n,q}(f;x)-f(x)| \\
&\le \Omega_m(f,\delta)\left(K_m(q)(1+x^m)+K^1_m(q)\frac{1}{\delta}\sqrt{\frac{2(1+q)}{(1+\sqrt{q})[n]_q}}(1+x^m)(1+x)\right) \\
&=\Omega_m(f,\delta)\left(K_m(q)(1+x^m)+K^1_m(q)K_1\frac{1}{\delta}\sqrt{\frac{2(1+q)}{(1+\sqrt{q})[n]_q}}(1+x^{m+1})\right)
\end{aligned}
$$

where

$$K_1=\sup_{x\ge 0}\frac{1+x^m+x+x^{m+1}}{1+x^{m+1}}.$$

If we take $\delta=\sqrt{\frac{2(1+q)}{(1+\sqrt{q})[n]_q}}$, then from the above inequality we obtain the desired result. □

4.3.3 Voronovskaja type theorem

Next, we prove the Voronovskaja-type result for L^*_{n,q_n}.

Theorem 4.24. *Assume that* $q_n\in(0,1)$, $q_n\to 1$ *and* $q_n^n\to a$ $(0\le a<1)$ *as* $n\to\infty$. *For any* $f\in C^*_2[0,\infty)$ *such that* $f',f''\in C^*_2[0,\infty)$ *the following equality holds*

$$\lim_{n\to\infty}[n]_{q_n}\left(L^*_{n,q_n}(f;x)-f(x)\right)=\frac{(2-a)x+1}{2}\left(xf''(x)-f'(x)\right)$$

uniformly on any $[0,A]$, $A>0$.

Proof. Let $f,f',f''\in C^*_2[0,\infty)$ and $x\in[0,\infty)$ be fixed. By the Taylor's formula we may write

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}(t-x)^2f''(x)+r(t;x)(t-x)^2, \tag{4.24}$$

where $r(t;x)$ is the Peano form of remainder, $r(.;x)\in C^*_2[0,\infty)$ and $\lim_{t\to x}r(t;x)=0$. Applying L^*_{n,q_n} to (4.24) we obtain

$$
\begin{aligned}
[n]_{q_n}\left(L^*_{n,q_n}(f;x)-f(x)\right)&=[n]_{q_n}L^*_{n,q_n}(e_1-e_0x;x)f'(x)\\
&+\frac{1}{2}[n]_{q_n}L^*_{n,q_n}\left((e_1-e_0x)^2;x\right)f''(x)+[n]_{q_n}L^*_{n,q_n}\left(r(t;x)(t-x)^2;x\right).
\end{aligned}
$$

By the Cauchy-Schwartz inequality, we have

$$L^*_{n,q_n}\left(r(t;x)(t-x)^2;x\right)\le\sqrt{L^*_{n,q_n}\left(r^2(t;x);x\right)}\sqrt{L^*_{n,q_n}\left((t-x)^4;x\right)}. \tag{4.25}$$

Observe that $r^2(x;x) = 0$ and $r^2(t;x) \in C_2^*[0,\infty)$. Then it follows from Theorem 4.21 that

$$\lim_{n\to\infty} L_{n,q_n}^*\left(r^2(t;x);x\right) = r^2(x;x) = 0 \tag{4.26}$$

uniformly with respect to $x \in [0,A]$. Now from (4.25), (4.26) and Lemma 4.19, we immediately get

$$\lim_{n\to\infty}[n]_{q_n} L_{n,q_n}^*\left(r(t;x)(t-x)^2;x\right) = 0.$$

This completes the proof. □

Chapter 5

On the q-Bernstein-Kantorovich operators with shifted knots

5.1 Introduction

In 2013, Mahmudov [74] introduced a q-type generalization of Bernstein-Kantorovich operators as follows:

$$B^*_{n,q}(f;x) = \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 f\left(\frac{[k]_q + q^k t}{[n+1]_q}\right) d_q t, \tag{5.1}$$

where

$$p_{n,k}(q,x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)_q^{n-k}, \;\; (1-x)_q^n = \prod_{s=0}^{n-1}(1-q^s x).$$

It can be seen that for $q \to 1^-$ the q-Bernstein-Kantorovich operator becomes the classical Bernstein-Kantorovich operator (5.1).

İçöz [49] introduced a Kantorovich type generalization of Bernstein-Stancu polynomials as follows:

$$\begin{aligned} &S^*_{n,\alpha,\beta}(f;x) \\ &= (n+\beta_1+1)\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} \binom{n}{r} \left(x - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n-r} \int_{\frac{r+\alpha_1}{n+\beta_1+1}}^{\frac{r+\alpha_1+1}{n+\beta_1+1}} f(s)ds, \end{aligned} \tag{5.2}$$

where $\frac{\alpha_2}{n+\beta_2} \le x \le \frac{n+\alpha_2}{n+\beta_2}$ and α_k, β_k $(k = 1,2)$ are positive real numbers provided $0 \le \alpha_1 \le \alpha_2 \le \beta_1 \le \beta_2$. It is clear that for $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$ then these polynomials turn into the Bernstein-Kantorovich operators.

5.2 Operators and some auxiliary results

Motivated by the work done by Mahmodov [74] and İçöz [49], we construct a Kantorovich type q-Bernstein-Stancu type polynomials as follows:

$$K_{n,q}^{(\alpha,\beta)}(f;x)=\left(\frac{[n]_q+\beta_2}{[n]_q}\right)^n\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\left(x-\frac{\alpha_2}{[n]_q+\beta_2}\right)_q^k\left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}-x\right)_q^{n-k}$$
$$\int_0^1 f\left(\frac{[k]_q+q^kt+\alpha_1}{[n+1]_q+\beta_1}\right)d_qt, \tag{5.3}$$

where $\frac{\alpha_2}{[n]_q+\beta_2}\leq x\leq\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}$ and α_k,β_k $(k=1,2)$ are positive real numbers provided $0\leq\alpha_1\leq\alpha_2\leq\beta_1\leq\beta_2$. If we put for $\alpha_1=\alpha_2=\beta_1=\beta_2=0$ in (5.3) then these polynomials turn into the Bernstein-Kantorovich operators (5.1) introduced by Mahmudov. Throughout this paper, $\|.\|$ denotes the sup-norm on C[0,1].

The aim of this chapter is to study some approximation properties of Kantorovich type q-Bernstein-Stancu operators defined by (5.3). First, we prove the basic convergence of the introduced operators and also obtain the rate of convergence by these operators in terms of the modulus of continuity. Further, we study local approximation property and Voronovskaja type theorem for the said operators. With the help of the Matlab we have given an example to show the convergence of operators to a function.

Lemma 5.1. *Let $K_{n,q}^{(\alpha,\beta)}(f;x)$ be given by (5.3). Then the following properties hold:*

(i) $K_{n,q}^{(\alpha,\beta)}(1;x)=1$;

(ii) $K_{n,q}^{(\alpha,\beta)}(t;x)=\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}\frac{2q}{[2]_q}\left(x-\frac{\alpha_2}{[n]_q+\beta_2}\right)+\frac{1}{[n+1]_q+\beta_1}\left(\alpha_1+\frac{1}{[2]_q}\right)$;

$$(iii)\ K_{n,q}^{(\alpha,\beta)}(t^2;x)=\frac{q[n-1]_q}{[n]_q}\left(1+\frac{(q-1)^2}{[3]_q}+\frac{2(q-1)}{[2]_q}\right)\left(\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}\right)^2\left(x-\frac{\alpha_2}{[n]_q+\beta_2}\right)_q^2$$
$$+\left(1+\frac{q^2-1}{[3]_q}+(2\alpha_1+1)\frac{2q}{1+q}\right)\left(\frac{[n]_q+\beta_2}{([n+1]_q+\beta_1)^2}\right)\left(x-\frac{\alpha_2}{[n]_q+\beta_2}\right)$$
$$+\frac{1}{([n+1]_q+\beta_1)^2}\left(\alpha_1^2+\frac{2\alpha_1}{[2]_q}+\frac{1}{[3]_q}\right).$$

Proof. (iii) For $f(t)=t^2$, in view of (5.3) using $q^k=1+(q-1)[k]_q$, we have

$$K_{n,q}^{(\alpha,\beta)}(t^2;x)$$
$$=\left(\frac{[n]_q+\beta_2}{[n]_q}\right)^n\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\left(x-\frac{\alpha_2}{[n]_q+\beta_2}\right)_q^k\left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}-x\right)_q^{n-k}\int_0^1\left(\frac{[k]_q+q^kt+\alpha_1}{[n+1]_q+\beta_1}\right)^2d_qt$$

$$= \frac{1}{([n+1]_q+\beta_1)^2}\left(\frac{[n]_q+\beta_2}{[n]_q}\right)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(x-\frac{\alpha_2}{[n]_q+\beta_2}\right)_q^k \left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}-x\right)_q^{n-k}$$
$$\int_0^1 \left([k]_q+(1+(q-1)[k]_q)t+\alpha_1\right)^2 d_q t$$
$$= \frac{1}{([n+1]_q+\beta_1)^2}\left(\frac{[n]_q+\beta_2}{[n]_q}\right)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(x-\frac{\alpha_2}{[n]_q+\beta_2}\right)_q^k \left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}-x\right)_q^{n-k}$$
$$\left\{\left(1+\frac{(q-1)^2}{[3]_q}+\frac{2(q-1)}{[2]_q}\right)[k]_q^2\right.$$
$$\left.+\left(\frac{2(q-1)}{[3]_q}+2\alpha_1+\frac{2\alpha_1(q-1)}{[2]_q}+\frac{2}{[2]_q}\right)[k]_q+\alpha_1^2+\frac{2\alpha_1}{[2]_q}+\frac{1}{[3]_q}\right\}$$
$$= \frac{1}{([n+1]_q+\beta_1)^2}\left\{\left(1+\frac{(q-1)^2}{[3]}+\frac{2(q-1)}{[2]_q}\right)\left(\frac{[n]_q+\beta_2}{[n]_q}\right)^n\right.$$
$$\sum_{k=0}^{n-1}[n]_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \left(x-\frac{\alpha_2}{[n]_q+\beta_2}\right)_q^{k+1} \left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}-x\right)_q^{n-k-1} [k+1]_q$$
$$+\left(\frac{2(q-1)}{[3]_q}+2\alpha_1+\frac{2\alpha_1(q-1)}{[2]_q}+\frac{2}{[2]_q}\right)\left(\frac{[n]_q+\beta_2}{[n]_q}\right)^n$$
$$\left.\sum_{k=0}^{n-1}[n]_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \left(x-\frac{\alpha_2}{[n]_q+\beta_2}\right)_q^{k+1} \left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}-x\right)_q^{n-k-1} + \left(\alpha_1^2+\frac{2\alpha_1}{[2]_q}+\frac{1}{[3]_q}\right)\right\}$$
$$= \frac{1}{([n+1]_q+\beta_1)^2}\left\{\left(1+\frac{(q-1)^2}{[3]_q}+\frac{2(q-1)}{[2]_q}\right)\left(\frac{[n]_q+\beta_2}{[n]_q}\right)^n\right.$$
$$\sum_{k=1}^{n-1}[n]_q \frac{[n-1]_q}{[k]_q} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q \left(x-\frac{\alpha_2}{[n]_q+\beta_2}\right)_q^{k+1} \left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}-x\right)_q^{n-k-1} q[k]_q$$
$$+\left(1+\frac{(q-1)^2}{[3]_q}+\frac{2(q-1)}{[2]_q}+\frac{2(q-1)}{[3]_q}+2\alpha_1+\frac{2\alpha_1(q-1)}{[2]_q}+\frac{2}{[2]_q}\right)\left(\frac{[n]_q+\beta_2}{[n]_q}\right)^n$$
$$\left.\sum_{k=0}^{n-1}[n]_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \left(x-\frac{\alpha_2}{[n]_q+\beta_2}\right)_q^{k+1} \left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}-x\right)_q^{n-k-1} + \left(\alpha_1^2+\frac{2\alpha_1}{[2]_q}+\frac{1}{[3]_q}\right)\right\}$$
$$= \frac{q[n-1]_q}{[n]_q}\left(1+\frac{(q-1)^2}{[3]_q}+\frac{2(q-1)}{[2]_q}\right)\left(\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}\right)^2\left(x-\frac{\alpha_2}{[n]_q+\beta_2}\right)_q^2$$
$$+\left(1+\frac{q^2-1}{[3]_q}+(2\alpha_1+1)\frac{2q}{[2]_q}\right)\left(\frac{[n]_q+\beta_2}{([n+1]_q+\beta_1)^2}\right)\left(x-\frac{\alpha_2}{[n]_q+\beta_2}\right)$$
$$\frac{1}{([n+1]_q+\beta_1)^2}\left(\alpha_1^2+\frac{2\alpha_1}{[2]_q}+\frac{1}{[3]_q}\right)$$

□

Lemma 5.2. *For all* $x \in \left[\frac{\alpha_2}{[n]_q+\beta_2}, \frac{[n]_q+\alpha_2}{[n]_q+\beta_2}\right]$, *we have*

$$
\begin{aligned}
&K_{n,q}^{(\alpha,\beta)}\left((t-x)^2;x\right)\\
&\leq \frac{2q^2(2q+1)}{[2]_q[3]_q}\frac{[n]_q([n]_q+\alpha_2)}{([n+1]_q+\beta_1)^2}+\frac{q}{1+q}\left(\frac{3+5q+4q^2}{1+q+q^2}+4\alpha_1\right)\frac{[n]_q}{([n+1]_q+\beta_1)^2}\\
&-\frac{2}{1+q}\frac{(2q[n]_q+2\alpha_1+1)([n]_q+\alpha_2)}{([n+1]_q+\beta_1)([n]_q+\beta_2)}+\left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}\right)^2+\left(\frac{1+\alpha_1}{[n+1]_q+\beta_1}\right)^2.
\end{aligned}
$$

Proof. From Lemma 5.1, we have

$$
\begin{aligned}
&K_{n,q}^{(\alpha,\beta)}\left((t-x)^2;x\right)\\
&=\left\{\frac{q[n-1]_q}{[n]_q}\left(1+\frac{(q-1)^2}{[3]_q}+\frac{2(q-1)}{[2]_q}\right)\left(\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}\right)^2\right.\\
&\left.-\frac{4q}{1+q}\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}+1\right\}x^2\\
&+\left\{\left(1+\frac{q^2-1}{[3]_q}+(2\alpha_1+1)\frac{2q}{1+q}\right)\frac{[n]_q+\beta_2}{([n+1]_q+\beta_1)^2}\right.\\
&-\frac{q[n-1]_q}{[n]_q}\frac{[2]_q\alpha_2}{[n]_q+\beta_2}\left(1+\frac{(q-1)^2}{[3]_q}+\frac{2(q-1)}{[2]_q}\right)\left(\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}\right)^2\\
&\left.+\frac{2q}{1+q}\frac{2\alpha_2}{[n]_q+\beta_2}\left(\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}\right)-\frac{2}{[n+1]_q+\beta_1}\left(\alpha_1+\frac{1}{[2]_q}\right)\right\}x\\
&+\frac{q^2[n-1]_q}{[n]_q}\left(1+\frac{(q-1)^2}{[3]_q}+\frac{2(q-1)}{[2]_q}\right)\left(\frac{\alpha_2}{[n+1]_q+\beta_1}\right)^2\\
&-\left(1+\frac{q^2-1}{[3]_q}+(2\alpha_1+1)\frac{2q}{1+q}\right)\frac{\alpha_2}{([n+1]_q+\beta_1)^2}\\
&+\frac{1}{([n+1]_q+\beta_1)^2}\left(\alpha_1^2+\frac{2\alpha_1}{[2]_q}+\frac{1}{[3]_q}\right).
\end{aligned}
$$

By using the monotonicity of positive linear operators $K_{n,q}^{(\alpha,\beta)}$ over $\left[\frac{\alpha_2}{[n]_q+\beta_2}, \frac{[n]_q+\alpha_2}{[n]_q+\beta_2}\right]$, we have

$$
\begin{aligned}
&K_{n,q}^{(\alpha,\beta)}\left((t-x)^2;x\right)\\
&\leq\left(1-\frac{1}{[n]_q}\right)\left(1+\frac{(q-1)^2}{[3]_q}+\frac{2(q-1)}{[2]_q}\right)\\
&\left\{\left(\frac{[n]_q+\alpha_2}{[n+1]_q+\beta_1}\right)^2-\frac{[2]_q\alpha_2([n]_q+\alpha_2)}{([n+1]_q+\beta_1)^2}+\frac{q\alpha_2^2}{([n+1]_q+\beta_1)^2}\right\}\\
&+\left(1+\frac{q^2-1}{[3]_q}+(2\alpha_1+1)\frac{2q}{1+q}\right)\frac{[n]_q}{([n+1]_q+\beta_1)^2}+\left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4q}{1+q}\frac{\alpha_2([n]_q+\alpha_2)}{([n+1]_q+\beta_1)([n]_q+\beta_2)}-2\left(\alpha_1+\frac{1}{[2]_q}\right)\frac{([n]_q+\alpha_2)}{([n+1]_q+\beta_1)([n]_q+\beta_2)}\\
&-\frac{4q}{1+q}\frac{([n]_q+\alpha_2)^2}{([n+1]_q+\beta_1)([n]_q+\beta_2)}+\frac{1}{([n+1]_q+\beta_1)^2}\left(\alpha_1^2+\frac{2\alpha_1}{[2]_q}+\frac{1}{[3]_q}\right)\\
&=\left(1-\frac{1}{[n]_q}\right)\frac{2q^2(2q+1)}{[2]_q[3]_q}\frac{[n]_q^2+(1-q)[n]_q\alpha_2}{([n+1]_q+\beta_1)^2}\\
&+\frac{q}{1+q}\left(\frac{3+5q+4q^2}{1+q+q^2}+4\alpha_1\right)\frac{[n]_q}{([n+1]_q+\beta_1)^2}+\left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}\right)^2\\
&+\frac{2(2q\alpha_2-[2]_q\alpha_1-1)}{[2]_q}\frac{[n]_q+\alpha_2}{([n+1]_q+\beta_1)([n]_q+\beta_2)}\\
&-\frac{4q}{1+q}\frac{([n]_q+\alpha_2)^2}{([n+1]_q+\beta_1)([n]_q+\beta_2)}+\frac{1}{([n+1]_q+\beta_1)^2}\left(\alpha_1^2+\frac{2\alpha_1}{[2]_q}+\frac{1}{[3]_q}\right)\\
&\le\frac{2q^2(2q+1)}{[2]_q[3]_q}\frac{[n]_q([n]_q+\alpha_2)}{([n+1]_q+\beta_1)^2}+\frac{q}{1+q}\left(\frac{3+5q+4q^2}{1+q+q^2}+4\alpha_1\right)\frac{[n]_q}{([n+1]_q+\beta_1)^2}\\
&-\frac{2}{1+q}\frac{(2q[n]_q+2\alpha_1+1)([n]_q+\alpha_2)}{([n+1]_q+\beta_1)([n]_q+\beta_2)}+\left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}\right)^2+\left(\frac{1+\alpha_1}{[n+1]_q+\beta_1}\right)^2
\end{aligned}
$$

which is the required result. □

Lemma 5.3. *Assume that* $0<q_n<1$, $q_n\to 1$ *and* $q_n^n\to a$ $(0\le a<1)$ *as* $n\to\infty$. *Then we have*

$$
\lim_{n\to\infty}[n]_{q_n}K_{n,q_n}^{(\alpha,\beta)}(t-x;x)=-\frac{1+a+2(\beta_1-\beta_2)}{2}x+\frac{1+2(\alpha_1-\alpha_2)}{2};
$$

$$
\lim_{n\to\infty}[n]_{q_n}K_{n,q_n}^{(\alpha,\beta)}((t-x)^2;x)=\left(a+2\beta_1-2\beta_2\right)x^2+x.
$$

Proof. To prove the lemma we use formulae for $K_{n,q_n}^{\alpha,\beta}(t;x)$ and $K_{n,q_n}^{\alpha,\beta}(t^2;x)$ given in Lemma 5.1.

$$
\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n}K_{n,q_n}^{(\alpha,\beta)}(t-x;x)\\
&=\lim_{n\to\infty}\left(\frac{[n]_{q_n}}{[n+1]_{q_n}+\beta_1}\right)\frac{2q_n([n]_{q_n}+\beta_2)-(1+q_n)([n+1]_{q_n}+\beta_1)}{[2]_{q_n}}x\\
&-\alpha_2\lim_{n\to\infty}\left(\frac{[n]_{q_n}}{[n+1]_{q_n}+\beta_1}\right)\frac{2q_n}{1+q_n}+\lim_{n\to\infty}\left(\alpha_1+\frac{1}{[2]_{q_n}}\right)\frac{[n]_{q_n}}{[n+1]_{q_n}+\beta_1}\\
&=\lim_{n\to\infty}\frac{q_n^{n+1}-q_n^{n+2}-q_n+1+2q_n\beta_2(q_n-1)+\beta_1(1-q_n^2)}{q_n-1}\frac{x}{[2]_{q_n}}-\alpha_2+\alpha_1+\frac{1}{2}\\
&=-\frac{1+a+2(\beta_1-\beta_2)}{2}x+\frac{1+2(\alpha_1-\alpha_2)}{2}.
\end{aligned}
$$

$$
\lim_{n\to\infty}[n]_{q_n}K_{n,q_n}^{(\alpha,\beta)}((t-x)^2;x)
$$

$$
\begin{aligned}
&= \lim_{n\to\infty}[n]_{q_n}\left(K_{n,q_n}^{(\alpha,\beta)}(t^2;x) - x^2 - 2xK_{n,q_n}^{(\alpha,\beta)}(t-x;x)\right)\\
&= \lim_{n\to\infty}[n]_{q_n}\left\{\left(1-\frac{1}{[n]_{q_n}}\right)\left(1+\frac{(q_n-1)^2}{[3]_{q_n}}+\frac{2(q_n-1)}{[2]_{q_n}}\right)\left(\frac{[n]_{q_n}+\beta_2}{[n+1]_{q_n}+\beta_1}\right)^2-1\right\}x^2\\
&+ \lim_{n\to\infty}[n]_{q_n}\left\{-\left(1-\frac{1}{[n]_{q_n}}\right)\left(1+\frac{(q_n-1)^2}{[3]_{q_n}}+\frac{2(q_n-1)}{[2]_{q_n}}\right)\left(\frac{[n]_{q_n}+\beta_2}{[n+1]_{q_n}+\beta_1}\right)^2\frac{[2]_{q_n}\alpha_2}{[n]_{q_n}+\beta_2}\right.\\
&+ \left.\left(1+\frac{q_n^2-1}{[3]_{q_n}}+\frac{2q_n(2\alpha_1+1)}{1+q_n}\right)\frac{[n]_{q_n}+\beta_2}{\left([n+1]_{q_n}+\beta_1\right)^2}\right\}x\\
&+ \lim_{n\to\infty}[n]_{q_n}\left(1-\frac{1}{[n]_{q_n}}\right)\left(1+\frac{(q_n-1)^2}{[3]_{q_n}}+\frac{2(q_n-1)}{[2]_{q_n}}\right)\left(\frac{[n]_{q_n}+\beta_2}{[n+1]_{q_n}+\beta_1}\right)^2\frac{q_n\alpha_2^2}{([n]_{q_n}+\beta_2)^2}\\
&- \lim_{n\to\infty}[n]_{q_n}\left(1+\frac{q_n^2-1}{[3]_{q_n}}+\frac{2q_n(2\alpha_1+1)}{1+q_n}\right)\frac{[n]_{q_n}+\beta_2}{\left([n+1]_{q_n}+\beta_1\right)^2}\frac{\alpha_2}{[n]_{q_n}+\beta_2}\\
&- 2x\lim_{n\to\infty}[n]_{q_n}K_{n,q_n}^{(\alpha,\beta)}(t-x;x)\\
&= -2\alpha_2 x - x^2 + 2x + 2\alpha_1 x - 2x\left(-\frac{1+a+2(\beta_1-\beta_2)}{2}x+\frac{1+2(\alpha_1-\alpha_2)}{2}\right)\\
&= (a+2\beta_1-2\beta_2)x^2 + x.
\end{aligned}
$$

□

5.3 Korovkin type approximation

First we give the following theorem on convergence of $K_{n,q}^{(\alpha,\beta)}(f;x)$ to $f(x)$.

Theorem 5.4. *Let $q = q_n \in (0,1)$ be a sequence such that $q_n \to 1$ as $n \to \infty$ and f a continuous function on [0,1]. Then*

$$\lim_{n\to\infty}\max_{\frac{\alpha_2}{[n]_{q_n}+\beta_2}\le x\le\frac{[n]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}} \mid K_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)\mid = 0.$$

Proof. Taking into consideration the equalities in Lemma 5.1, for $v = 0,1,2$ we can write

$$\lim_{n\to\infty}\max_{\frac{\alpha_2}{[n]_{q_n}+\beta_2}\le x\le\frac{[n]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}}\left|K_{n,q_n}^{(\alpha,\beta)}(t^v;x)-x^v\right| = 0. \tag{5.4}$$

Now consider the sequence of operators

$$K_{n,q_n}^{*}(f;x) = \begin{cases} K_{n,q_n}^{(\alpha,\beta)}, & \text{if } \frac{\alpha_2}{[n]_{q_n}+\beta_2}\le x\le\frac{[n]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2};\\ f(x), & \text{if } x\in\left[0,\frac{\alpha_2}{[n]_{q_n}+\beta_2}\right]\cup\left[\frac{[n]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2},1\right].\end{cases}$$

Then obviously,

$$\|K^*_{n,q_n}f - f\| = \max_{\frac{\alpha_2}{[n]_{q_n}+\beta_2} \le x \le \frac{[n]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}} | K^{(\alpha,\beta)}_{n,q_n}(f;x) - f(x) | \tag{5.5}$$

and using (5.4) we obtain

$$\lim_{n\to\infty} \|K^*_{n,q_n}(t^\nu;x) - x^\nu\|_{C[0,1]} = 0, \;\; \nu = 0,1,2.$$

Applying the Korovkin theorem [60] (see also [2]) to the sequence of positive linear operators K^*_{n,q_n}, we obtain

$$\lim_{n\to\infty} \|K^*_{n,q_n}(f;x) - f(x)\|_{C[0,1]} = 0$$

for every continuous function f. Therefore (5.5) gives

$$\lim_{n\to\infty} \max_{\frac{\alpha_2}{[n]_{q_n}+\beta_2} \le x \le \frac{[n]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}} | K^{(\alpha,\beta)}_{n,q_n}(f;x) - f(x) | = 0$$

and thus the result is obtained. □

Example 5.1. *With the help of Matlab, we show comparisons and some illustrative graphics for the convergence of operators (5.3) to the function $f(x) = 1 - cos(4e^x)$ under different parameters.*

From figure 5.1-5.3, we can observe that as the value of n increases, Kantorovich type q-Bernstein-Stancu operators given by (5.3) converge towards the function. Similarly, as the value of q increases, convergence of operators to the function is shown in figure 5.4 with different values of parameters α_1, α_2, β_1, β_2, and n.

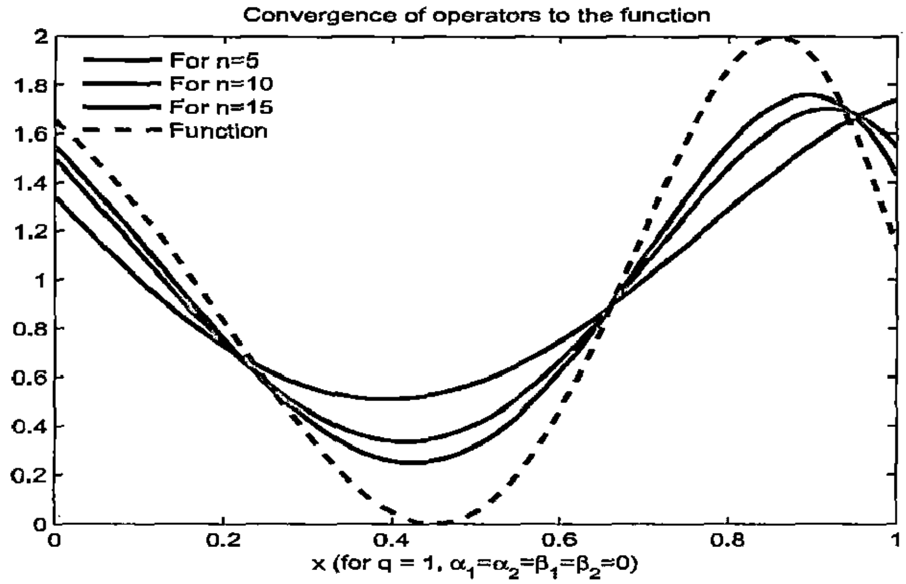

Figure 5.1:

We use modulus of continuity to give quantitative error estimates for the approximation by positive linear operators.

Theorem 5.5. *If* $f \in C[0,1]$ *and* $0 < q < 1$, *then*

$$\|K_{n,q}^{(\alpha,\beta)}(f;x) - f(x)\| \;\leq\; 2\omega(f,\delta_n),$$

where

$$\delta_n^2 = \frac{2q^2(2q+1)}{[2]_q[3]_q}\frac{[n]_q([n]_q+\alpha_2)}{([n+1]_q+\beta_1)^2} + \frac{q}{1+q}\left(\frac{3+5q+4q^2}{1+q+q^2}+4\alpha_1\right)\frac{[n]_q}{([n+1]_q+\beta_1)^2}$$
$$-\frac{2}{1+q}\frac{(2q[n]_q+2\alpha_1+1)([n]_q+\alpha_2)}{([n+1]_q+\beta_1)([n]_q+\beta_2)} + \left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}\right)^2 + \left(\frac{1+\alpha_1}{[n+1]_q+\beta_1}\right)^2.$$

Proof. From (1.6), for any $x, y \in [a,b]$, we have

$$|f(y) - f(x)| \;\leq\; \omega(f,\delta)\Big(\frac{(y-x)^2}{\delta^2}+1\Big).$$

Therefore, we get

$$|K_{n,q}^{(\alpha,\beta)}(f;x) - f(x)|$$
$$\leq K_{n,q}^{(\alpha,\beta)}(|f(t)-f(x)|;x) \leq \omega_f(\delta)\Big(1+\frac{1}{\delta^2}K_{n,q}^{(\alpha,\beta)}\big((t-x)^2;x\big)\Big).$$

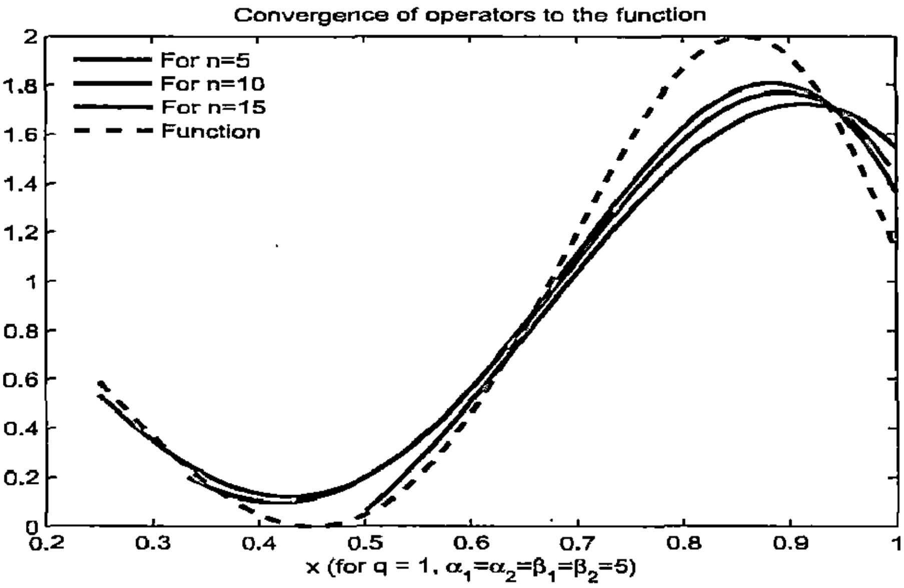

Figure 5.2:

By using Lemma 5.2, we can write

$$
\begin{aligned}
&|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| \\
&\leq \omega(f,\delta)\Bigg[1+\frac{1}{\delta^2}\Bigg\{\frac{2q^2(2q+1)}{[2]_q[3]_q}\frac{[n]_q([n]_q+\alpha_2)}{([n+1]_q+\beta_1)^2}+\frac{q}{1+q}\left(\frac{3+5q+4q^2}{1+q+q^2}+4\alpha_1\right)\frac{[n]_q}{([n+1]_q+\beta_1} \\
&-\frac{2}{1+q}\frac{(2q[n]_q+2\alpha_1+1)([n]_q+\alpha_2)}{([n+1]_q+\beta_1)([n]_q+\beta_2)}+\left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}\right)^2+\left(\frac{1+\alpha_1}{[n+1]_q+\beta_1}\right)^2\Bigg\}\Bigg].
\end{aligned}
$$

Choosing

$$
\begin{aligned}
\delta=\delta_n=&\Bigg\{\frac{2q^2(2q+1)}{[2]_q[3]_q}\frac{[n]_q([n]_q+\alpha_2)}{([n+1]_q+\beta_1)^2}+\frac{q}{1+q}\left(\frac{3+5q+4q^2}{1+q+q^2}+4\alpha_1\right)\frac{[n]_q}{([n+1]_q+\beta_1)^2} \\
&-\frac{2}{1+q}\frac{(2q[n]_q+2\alpha_1+1)([n]_q+\alpha_2)}{([n+1]_q+\beta_1)([n]_q+\beta_2)}+\left(\frac{[n]_q+\alpha_2}{[n]_q+\beta_2}\right)^2+\left(\frac{1+\alpha_1}{[n+1]_q+\beta_1}\right)^2\Bigg\}^{\frac{1}{2}},
\end{aligned}
$$

we have

$$
\|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)\| \;\leq\; 2\omega(f,\delta_n).
$$

Thus, we obtain the desired result. □

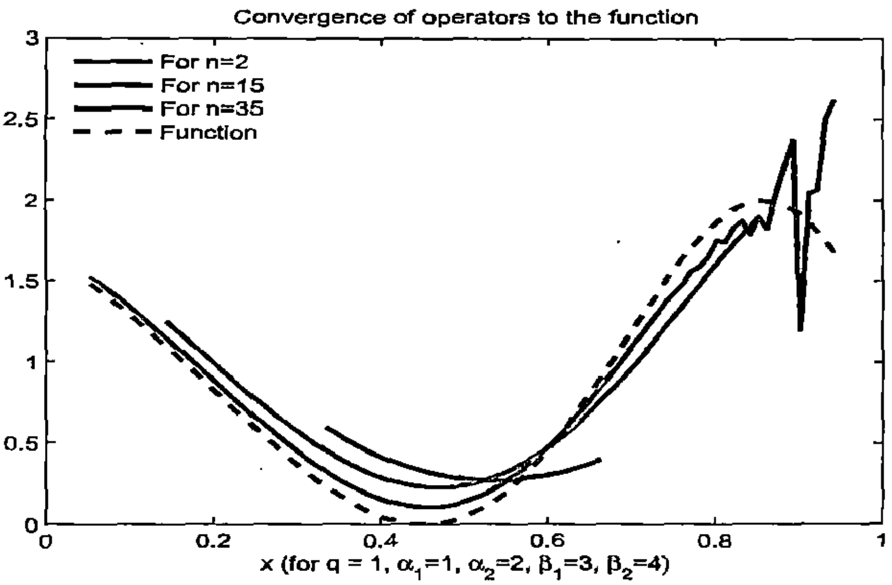

Figure 5.3:

5.4 Direct theorems

Theorem 5.6. *Let ω, ω_2 be the moduli of continuity given by the expression (1.3) and (1.4), respectively and let $f \in C[0,1]$ with $0<q<1$. Then for every $x \in \left[\frac{\alpha_2}{[n]_q+\beta_2}, \frac{[n]_q+\alpha_2}{[n]_q+\beta_2}\right]$, we have*

$$\left|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)\right| \le C\omega_2\left(f,\sqrt{\delta_n(x)}\right)+\omega\left(f,|(a_n-1)x+b_n|\right),$$

where $a_n=\frac{2q}{1+q}\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}$, $b_n=\frac{1}{[n+1]_q+\beta_1}\left(\alpha_1+\frac{1}{1+q}\right)-\frac{2q}{1+q}\frac{\alpha_2}{[n+1]_q+\beta_1}$ *and*

$$\delta_n(x)=\left\{\frac{1+2q+4q^2+5q^3}{1+2q+2q^2+q^3}\left(\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}\right)^2-\frac{2(3q+1)}{1+q}\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}+2\right\}x^2$$
$$+\left\{\left(\frac{5+7q+6q^2}{1+q+q^2}+\frac{2q^2(2q+1)\alpha_2}{(1+q+q^2)[n]_q}+4\alpha_1\right)\frac{[n]_q+\beta_2}{([n+1]_q+\beta_1)^2}+2\frac{\alpha_2}{[n+1]_q+\beta_1}\right\}x$$
$$+\frac{q^2(2q+1)}{1+q+q^2}\left(\frac{\alpha_2}{[n+1]_q+\beta_1}\right)^2-\frac{q}{1+q}\left(\frac{3+5q+4q^2}{1+q+q^2}+4\alpha_1\right)\frac{\alpha_2}{([n+1]_q+\beta_1)^2}+2\left(\frac{1+\alpha_1}{[n+1]_q+\beta_1}\right)^2.$$

Proof. Let

$$\bar{K}_{n,q}^{(\alpha,\beta)}(f;x)=K_{n,q}^{(\alpha,\beta)}(f;x)+f(x)-f(a_nx+b_n),$$

where $f\in C[0,1]$, $a_n=\frac{2q}{1+q}\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}$ and $b_n=\frac{1}{[n+1]_q+\beta_1}\left(\alpha_1+\frac{1}{1+q}\right)-\frac{2q}{1+q}\frac{\alpha_2}{[n+1]_q+\beta_1}$.

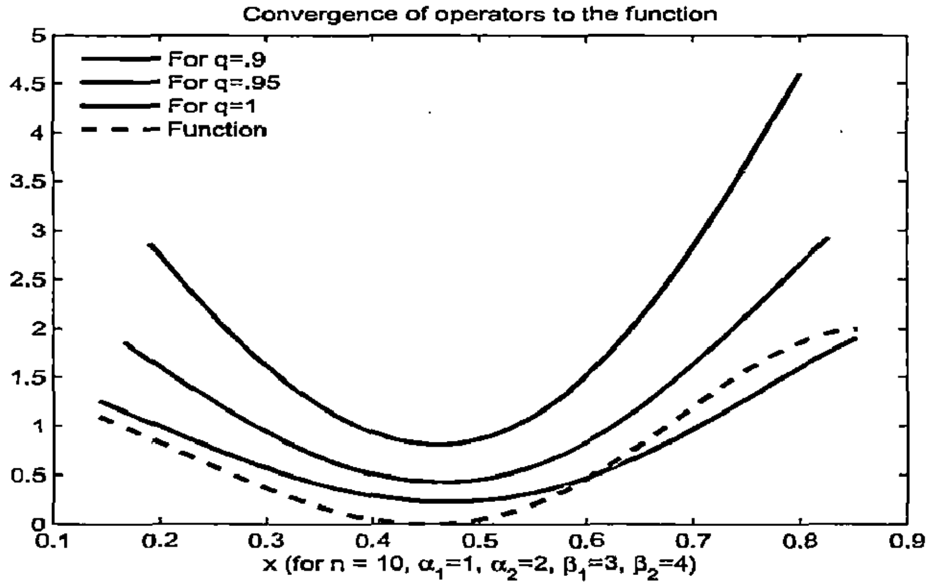

Figure 5.4:

Using the Taylor's formula

$$g(t) = g(x) + g'_x(t-x) + \int_x^t (t-s)g''(s)ds, \quad g \in C^2[0,1],$$

we have

$$\tilde{K}_{n,q}^{(\alpha,\beta)}(g;x) = g(x) + K_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-s)g''(s)ds; x\right) - \int_x^{a_nx+b_n} (a_nx+b_n-s)g''(s)ds, \quad g \in C^2[0,1]$$

Hence

$$\begin{aligned}
&\left|\tilde{K}_{n,q}^{(\alpha,\beta)}(g;x) - g(x)\right| \\
&\leq K_{n,q}^{(\alpha,\beta)}\left(\left|\int_x^t (t-s)g''(s)ds\right|; x\right) + \left|\int_x^{a_nx+b_n} |a_nx+b_n-s|\,|g''(s)|ds\right| \\
&\leq K_{n,q}^{(\alpha,\beta)}\left((t-x)^2; x\right)\|g''\| + (a_nx+b_n-x)^2\|g''\| \\
&= \left\{ \left(1 - \frac{1}{[n]_q}\right) \frac{2q^2(2q+1)}{[2]_q[3]_q} \left(\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}\right)^2 \right. \\
&\left. - \frac{4q}{1+q}\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1} + 1 + \left(\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1} - 1\right)^2 \right\} x^2 \\
&+ \left\{ \frac{q}{1+q}\left(\frac{3+5q+4q^2}{1+q+q^2} + 4\alpha_1\right) \frac{[n]_q+\beta_2}{([n+1]_q+\beta_1)^2} \right.
\end{aligned}$$

$$
\begin{aligned}
&-\left(1-\frac{1}{[n]_q}\right)\frac{\alpha_2}{[n]_q+\beta_2}\frac{2q^2(2q+1)}{[3]_q}\left(\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}\right)^2\\
&+\frac{4q}{1+q}\frac{\alpha_2}{[n+1]_q+\beta_1}-\frac{2}{[n+1]_q+\beta_1}\left(\alpha_1+\frac{1}{[2]_q}\right)+2\left(\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}-1\right)\left(\frac{1+\alpha_1}{[n+1]_q+\beta_1}\right)\Bigg\}x\\
&q\left(1-\frac{1}{[n]_q}\right)\frac{2q^2(2q+1)}{[2]_q[3]_q}\left(\frac{\alpha_2}{[n+1]+\beta_1}\right)^2-\frac{q}{1+q}\left(\frac{3+5q+4q^2}{1+q+q^2}+4\alpha_1\right)\frac{\alpha_2}{([n+1]_q+\beta_1)^2}\\
&+\frac{1}{([n+1]+\beta_1)^2}\left(\alpha_1^2+\frac{2\alpha_1}{[2]_q}+\frac{1}{[3]_q}\right)+\left(\frac{1+\alpha_1}{[n+1]_q+\beta_1}\right)^2\\
&\leq\left\{\frac{1+2q+4q^2+5q^3}{1+2q+2q^2+q^3}\left(\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}\right)^2-\frac{2(3q+1)}{1+q}\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1}+2\right\}x^2\\
&+\left\{\left(\frac{5+7q+6q^2}{1+q+q^2}+\frac{2q^2(2q+1)\alpha_2}{(1+q+q^2)[n]_q}+4\alpha_1\right)\frac{[n]_q+\beta_2}{([n+1]_q+\beta_1)^2}+2\frac{\alpha_2}{[n+1]_q+\beta_1}\right\}x\\
&+\frac{q^2(2q+1)}{1+q+q^2}\left(\frac{\alpha_2}{[n+1]_q+\beta_1}\right)^2\\
&-\frac{q}{1+q}\left(\frac{3+5q+4q^2}{1+q+q^2}+4\alpha_1\right)\frac{\alpha_2}{([n+1]_q+\beta_1)^2}+2\left(\frac{1+\alpha_1}{[n+1]_q+\beta_1}\right)^2\\
&=\delta_n(x)\|g''\|. \qquad (5.6)
\end{aligned}
$$

Using (5.6) and the uniform boundedness of $\bar{K}_{n,q}^{(\alpha,\beta)}$, we get

$$
\begin{aligned}
&\left|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)\right|\\
&\leq\left|\bar{K}_{n,q}^{(\alpha,\beta)}(f-g;x)\right|+\left|\bar{K}_{n,q}^{(\alpha,\beta)}(g;x)-g(x)\right|+\left|f(x)-g(x)\right|+\left|f(a_nx+b_n)-f(x)\right|\\
&\leq 4\|f-g\|+\delta_n(x)\|g''\|+\omega\big(f,|(a_n-1)x+b_n|\big).
\end{aligned}
$$

Taking the infimum on the right hand side over all $g\in C^2[0,1]$ and using (1.8) and (1.9), we obtain

$$
\left|K_{n,q}^{(\alpha,\beta)}(f;x)-f(x)\right|\leq C\omega_2\big(f,\sqrt{\delta_n(x)}\big)+\omega\big(f,|(a_n-1)x+b_n|\big).
$$

This completes the proof. □

Corollary 5.7. *Assume that $q_n\in(0,1)$, $q_n\to 1$ as $n\to\infty$. For any $f\in C^2[0,1]$ we have*

$$
\lim_{n\to\infty}\|K_{n,q_n}^{(\alpha,\beta)}(f)-f\|=0.
$$

Furthermore, we estimate the rate of convergence for smooth functions. For this reason, we now state following general estimate theorem obtained by Shisha and Mond [98] in terms of modulus of continuity.

Theorem 5.8. *Let $[c,d] \subseteq [a,b]$ and $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators such that*

$$L_n : C[a,b] \to C[c,d].$$

If $f' \in C[a,b]$ and $x \in [c,d]$, then we have

$$\begin{aligned}
&|L_n(f;x) - f(x)| \\
&\leq |f(x)||L_n(1;x) - 1| + |f'(x)||L_n(t-x;x)| + \sqrt{L_n((t-x)^2;x)} \\
&\quad \times \left\{\sqrt{L_n(1;x)} + \frac{1}{\delta}\sqrt{L_n((t-x)^2;x)}\right\}\omega(f',\delta),
\end{aligned}$$

where ω is the modulus of continuity of the function f defined by (1.3).

Theorem 5.9. *For any $f \in C^1[0,1]$ and each $x \in \left[\frac{\alpha_2}{[n]_q+\beta_2}, \frac{[n]_q+\alpha_2}{[n]_q+\beta_2}\right]$, we have*

$$\begin{aligned}
&|K_{n,q}^{(\alpha,\beta)} - f(x)| \\
&\leq \left|\left(\frac{2q}{1+q}\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1} - 1\right)x + \frac{1+\alpha_1+q\alpha_1-2q\alpha_2}{(1+q)([n+1]_q+\beta_1)}\right||f'(x)| + 2\sqrt{\delta_n(x)}\omega(f',\sqrt{\delta_n(x)}).
\end{aligned}$$

Proof. In view of Lemma 5.1, Lemma 5.2 & Theorem 5.8, and if we choose $\delta = \sqrt{\delta_n(x)} = \sqrt{K_{n,q}^{(\alpha,\beta)}((t-x)^2;x)}$, we have

$$\begin{aligned}
|K_{n,q}^{(\alpha,\beta)} - f(x)| &\leq \left|\left(\frac{2q}{1+q}\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1} - 1\right)x + \frac{1+\alpha_1+q\alpha_1-2q\alpha_2}{(1+q)([n+1]_q+\beta_1)}\right||f'(x)| \\
&+ \sqrt{K_{n,q}^{(\alpha,\beta)}((t-x)^2;x)}\left(1 + \frac{1}{\delta}\sqrt{K_{n,q}^{(\alpha,\beta)}((t-x)^2;x)}\right)\omega(f',\delta) \\
&= \left|\left(\frac{2q}{1+q}\frac{[n]_q+\beta_2}{[n+1]_q+\beta_1} - 1\right)x + \frac{1+\alpha_1+q\alpha_1-2q\alpha_2}{(1+q)([n+1]_q+\beta_1)}\right||f'(x)| + 2\sqrt{\delta_n(x)}\omega(f',\sqrt{\delta_n(x)}).
\end{aligned}$$

□

5.5 Voronovskaja type Theorem

Next we prove Voronovskaja type result for Kantorovich type q-Bernstein-Stancu operators.

Theorem 5.10. *Assume that $q = q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a$ $(0 \leq a < 1)$ as $n \to \infty$. For any $f \in C^2[0,1]$ the following equality holds*

$$\lim_{n\to\infty}[n]_{q_n}\left(K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right)$$

$$= f'(x)\left(-\frac{1+a+2(\beta_1-\beta_2)}{2}x+\frac{1+2(\alpha_1-\alpha_2)}{2}\right)+\frac{1}{2}f''(x)\left((a+2\beta_1-2\beta_2)x^2+x\right)$$

uniformly on $x \in \left[\frac{\alpha_2}{[n]_{q_n}+\beta_2}, \frac{[n]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}\right]$.

Proof. Let $f \in C^2[0,1]$ and $x \in [0,1]$ be fixed. By the Taylor's formula we may write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + r(t;x)(t-x)^2, \tag{5.7}$$

where $r(t;x)$ is the Peano form of remainder, $r(.;x) \in C[0,1]$ and $\lim_{t\to x} r(t,x) = 0$. Applying $K_{n,q_n}^{(\alpha,\beta)}(f;x)$ on both sides of (5.7), we obtain

$$\begin{aligned}
&[n]_{q_n}\left(K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right)\\
&= f'(x)[n]_{q_n}K_{n,q_n}^{(\alpha,\beta)}\left((t-x);x\right)+\frac{1}{2}f''(x)[n]_{q_n}K_{n,q_n}^{(\alpha,\beta)}\left((t-x)^2;x\right)\\
&+ [n]_{q_n}K_{n,q_n}^{(\alpha,\beta)}\left(r(t;x)(t-x)^2;x\right).
\end{aligned}$$

By the Cauchy-Schwartz inequality, we have

$$K_{n,q_n}^{(\alpha,\beta)}\left(r(t;x)(t-x)^2;x\right) \leq \sqrt{K_{n,q_n}^{(\alpha,\beta)}\left(r^2(t;x);x\right)}\sqrt{K_{n,q_n}^{(\alpha,\beta)}\left((t-x)^4;x\right)}. \tag{5.8}$$

Observe that $r^2(x,x) = 0$ and $r^2(.;x) \in C[0,1]$. Then it follows from Corollary 5.7 that

$$\lim_{n\to\infty} K_{n,q_n}^{(\alpha,\beta)}\left(r^2(t;x);x\right) = r^2(x,x) = 0 \tag{5.9}$$

uniformly with respect to $x \in \left[\frac{\alpha_2}{[n]_{q_n}+\beta_2}, \frac{[n]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}\right]$. Now from (5.8), (5.9) and Lemma 5.3, we get immediately

$$\lim_{n\to\infty}[n]_{q_n}K_{n,q_n}^{(\alpha,\beta)}\left(r(t;x)(t-x)^2;x\right) = 0.$$

The proof is completed. □

Now we give the rate of convergence of the operators $K_{n,q}^{(\alpha,\beta)}$ in terms of the elements of the usual Lipschitz class $Lip_M(\alpha)$ (for Def. see section 4.2.2).

Theorem 5.11. *Let* $q = q_n \in (0,1)$ *such that* $\lim_{n\to\infty} q_n = 1$. *Then for each* $f \in Lip_M(\alpha)$ *we have*

$$\|K_{n,q_n}^{(\alpha,\beta)} - f(x)\| \leq M\delta_n^{\alpha},$$

where $\|.\|$ *is the supremum norm over* $\left[\frac{\alpha_2}{[n]_{q_n}+\beta_2}, \frac{[n]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}\right]$ *and*

$$\begin{aligned}
&\|K_{n,q_n}^{(\alpha,\beta)}(f) - f\| \\
&\leq M\Bigg[\frac{2q_n^2(2q_n+1)}{[2]_{q_n}[3]_{q_n}}\frac{[n]_{q_n}([n]_{q_n}+\alpha_2)}{([n+1]_{q_n}+\beta_1)^2} + \frac{q_n}{1+q_n}\left(\frac{3+5q_n+4q_n^2}{1+q_n+q_n^2}+4\alpha_1\right)\frac{[n]_{q_n}}{([n+1]_{q_n}+\beta_1)^2} \\
&- \frac{2}{1+q_n}\frac{(2q_n[n]_{q_n}+2\alpha_1+1)([n]_{q_n}+\alpha_2)}{([n+1]_{q_n}+\beta_1)([n]_{q_n}+\beta_2)} + \left(\frac{[n]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}\right)^2 + \left(\frac{1+\alpha_1}{[n+1]_{q_n}+\beta_1}\right)^2\Bigg]^{\frac{\alpha}{2}}.
\end{aligned}$$

Proof. Let us denote

$$P_{n,k}^{(\alpha,\beta)}(x) = \left(\frac{[n]_{q_n}+\beta_2}{[n]_{q_n}}\right)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n} \left(x - \frac{\alpha_2}{[n]_{q_n}+\beta_2}\right)_{q_n}^k \left(\frac{[n]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2} - x\right)_{q_n}^{n-k}.$$

Then by the monotonicity of the operators $K_{n,q_n}^{(\alpha,\beta)}$, we can write

$$\begin{aligned}
|K_{n,q_n}^{(\alpha,\beta)} - f(x)| &\leq K_{n,q_n}^{(\alpha,\beta)}(|f(t)-f(x)|;x) \\
&\leq \sum_{k=0}^{n} P_{n,k}^{(\alpha,\beta)}(x)\int_0^1\left|f\left(\frac{[k]_{q_n}+q_n^k t+\alpha_1}{[n+1]_{q_n}+\beta_1}\right)-f(x)\right|d_{q_n}t \\
&\leq M\sum_{k=0}^{n} P_{n,k}^{(\alpha,\beta)}(x)\int_0^1\left|\frac{[k]_{q_n}+q_n^k t+\alpha_1}{[n+1]_{q_n}+\beta_1}-x\right|^{\alpha}d_{q_n}t.
\end{aligned}$$

On the other hand, by using the Hölder's inequality for integrals, we have

$$\begin{aligned}
&|K_{n,q_n}^{(\alpha,\beta)} - f(x)| \\
&\leq M\sum_{k=0}^{n} P_{n,k}^{(\alpha,\beta)}(x)\left\{\int_0^1\left(\frac{[k]_{q_n}+q_n^k t+\alpha_1}{[n+1]_{q_n}+\beta_1}-x\right)^2 d_{q_n}t\right\}^{\frac{\alpha}{2}}\left\{\int_0^1 1\, d_{q_n}t\right\}^{\frac{2-\alpha}{2}} \\
&= M\sum_{k=0}^{n}\left\{P_{n,k}^{(\alpha,\beta)}(x)\int_0^1\left(\frac{[k]_{q_n}+q_n^k t+\alpha_1}{[n+1]_{q_n}+\beta_1}-x\right)^2 d_{q_n}t\right\}^{\frac{\alpha}{2}} P_{n,k}^{(\alpha,\beta)}(x)^{\frac{2-\alpha}{2}}.
\end{aligned}$$

Now again applying the Hölder's inequality for the sum and taking into consideration Lemma 5.1 (i) and Lemma 5.2, we have

$$\begin{aligned}
&|K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)| \leq M\left(K_{n,q_n}^{(\alpha,\beta)}((t-x)^2;x)\right)^{\frac{\alpha}{2}}\left(K_{n,q_n}^{(\alpha,\beta)}(1;x)\right)^{\frac{2-\alpha}{2}} \\
&\leq M\Bigg[\Bigg\{\frac{q_n[n-1]_{q_n}}{[n]_{q_n}}\left(1+\frac{(q_n-1)^2}{[3]_{q_n}}+\frac{2(q_n-1)}{[2]_{q_n}}\right)\left(\frac{[n]_{q_n}+\beta_2}{[n+1]_{q_n}+\beta_1}\right)^2 \\
&- \frac{4q_n}{1+q_n}\frac{[n]_{q_n}+\beta_2}{[n+1]_{q_n}+\beta_1}+1\Bigg\}x^2
\end{aligned}$$

$$
\begin{aligned}
&+\Bigg\{\left(1+\frac{q_n^2-1}{[3]_{q_n}}+(2\alpha_1+1)\frac{2q_n}{1+q_n}\right)\frac{[n]_{q_n}+\beta_2}{([n+1]_{q_n}+\beta_1)^2}\\
&-\frac{q_n[n-1]_{q_n}}{[n]_{q_n}}\frac{[2]_{q_n}\alpha_2}{[n]_{q_n}+\beta_2}\left(1+\frac{(q_n-1)^2}{[3]_{q_n}}+\frac{2(q_n-1)}{[2]_{q_n}}\right)\left(\frac{[n]_{q_n}+\beta_2}{[n+1]_{q_n}+\beta_1}\right)^2\\
&+\frac{2q_n}{1+q_n}\frac{2\alpha_2}{[n]_{q_n}+\beta_2}\left(\frac{[n]_{q_n}+\beta_2}{[n+1]_{q_n}+\beta_1}\right)-\frac{2}{[n+1]_{q_n}+\beta_1}\left(\alpha_1+\frac{1}{[2]_{q_n}}\right)\Bigg\}x\\
&+\frac{q_n^2[n-1]_{q_n}}{[n]_{q_n}}\left(1+\frac{(q_n-1)^2}{[3]_{q_n}}+\frac{2(q_n-1)}{[2]_{q_n}}\right)\left(\frac{\alpha_2}{[n+1]_{q_n}+\beta_1}\right)^2\\
&-\left(1+\frac{q_n^2-1}{[3]_{q_n}}+(2\alpha_1+1)\frac{2q_n}{1+q_n}\right)\frac{\alpha_2}{([n+1]_{q_n}+\beta_1)^2}\\
&+\frac{1}{([n+1]_{q_n}+\beta_1)^2}\left(\alpha_1^2+\frac{2\alpha_1}{[2]_{q_n}}+\frac{1}{[3]_{q_n}}\right)\Bigg]^{\frac{\alpha}{2}}.
\end{aligned}
$$

Replacing x by $\frac{[n]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}$ implies that

$$
\begin{aligned}
&\|K_{n,q_n}^{(\alpha,\beta)}(f)-f\|\\
&\leq M\Bigg[\frac{2q_n^2(2q_n+1)}{[2]_{q_n}[3]_{q_n}}\frac{[n]_{q_n}([n]_{q_n}+\alpha_2)}{([n+1]_{q_n}+\beta_1)^2}+\frac{q_n}{1+q_n}\left(\frac{3+5q_n+4q_n^2}{1+q_n+q_n^2}+4\alpha_1\right)\frac{[n]_{q_n}}{([n+1]_{q_n}+\beta_1)^2}\\
&-\frac{2}{1+q_n}\frac{(2q_n[n]_{q_n}+2\alpha_1+1)([n]_{q_n}+\alpha_2)}{([n+1]_{q_n}+\beta_1)([n]_{q_n}+\beta_2)}+\left(\frac{[n]_{q_n}+\alpha_2}{[n]_{q_n}+\beta_2}\right)^2+\left(\frac{1+\alpha_1}{[n+1]_{q_n}+\beta_1}\right)^2\Bigg]^{\frac{\alpha}{2}}.
\end{aligned}
$$

Hence if we choose $\delta := \delta_n$, then we arrive at the desired result. □

Chapter 6

Approximation by (p, q)-analogue of Bernstein and Bernstein-Stancu operators

6.1 Introduction and preliminaries

During the last two decades, the applications of q-calculus emerged as a new area in the field of approximation theory. The rapid development of q-calculus has led to the discovery of various generalizations of Bernstein polynomials involving q-integers. Several researchers introduced and studied many positive linear operators based on q-integers, q-Bernstein basis, q-Beta basis, q-derivative and q-integrals etc.

Lupaş [67] was the first who introduced the q-analogue of the well known Bernstein polynomials and investigated its approximating and shape-preserving properties. For $f \in C[0,1]$, the positive linear operators $L_{n,q} : C[0,1] \to C[0,1]$, defined by

$$L_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) b^q_{n,k}(x), \tag{6.1}$$

where

$$b^q_{n,k}(x) = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k (1-x)^{n-k}}{\prod_{j=0}^{n-1}(1 - x + q^j x)}$$

are called Lupaş q-analogue of Bernstein polynomials.

Another q-generalization of the classical Bernstein polynomials is due to Phillips [85]. After that many generalizations of well-known positive linear operators, based on q-integers were introduced and studied by several authors. Recently the approximation properties have also been investigated for q-analogue various polynomials. For instance, see [38, 40, 41, 69, 72, 73, 80, 81].

In this chapter, we have introduced a new generalization of Bernstein and Bernstein-Stancu operators using two parameter quantum integers, i.e., (p, q)-integers. It seems that there are no other results in approximation theory by positive linear operators with the help of (p,q)-integers or (p,q)-calculus. We study the approximation properties based on Korovkin's approximation theorem and also establish some direct theorems for (p,q)-Bernstein and (p,q)-Bernstein-Stancu operators.

6.2 (p,q)-Bernstein operators

Now, we introduce (p,q)-analogue of Bernstein operators as

$$B_{n,p,q}(f;x)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k \prod_{s=0}^{n-k-1}(p^s-q^sx)\ f\left(\frac{[k]_{p,q}}{[n]_{p,q}}\right),\quad x\in[0,1]. \tag{6.2}$$

Note that for $p=1$, (p,q)- Bernstein operators given by (6.2) turn out to be q- Bernstein operators.

We note the following properties as follows:

6.2.1 Auxiliary results

We have the following basic results:

Lemma 6.1. *For $x\in[0,1]$, $0<q<p\leq 1$, we have*

(i) $B_{n,p,q}(1;x)=1;$

(ii) $B_{n,p,q}(t;x)=x;$

(iii) $B_{n,p,q}(t^2;x)=\frac{(px+1-x)_{p,q}^{n-1}x}{[n]_{p,q}}+\frac{q[n-1]_{p,q}x^2}{[n]_{p,q}};$

(iv) $B_{n,p,q}(t^3;x)=\frac{(p^2x+1-x)_{p,q}^{n-1}x}{[n]_{p,q}^2}+\frac{(2p+q)q[n-1]_{p,q}(px+1-x)_{p,q}^{n-2}x^2}{[n]_{p,q}^2}+\frac{q^3[n-1]_{p,q}[n-2]_{p,q}x^3}{[n]_{p,q}^2};$

(v) $B_{n,p,q}(t^4;x)=\frac{(p^3x+1-x)_{p,q}^{n-1}x}{[n]_{p,q}^3}+\frac{3p^2q[n-1]_{p,q}(px+1-x)_{p,q}^{n-2}x^2}{[n]_{p,q}^3}+\frac{3pq^2[n-1]_{p,q}(p^2x+1-x)_{p,q}^{n-2}x^2}{[n]_{p,q}^3}$
$+\frac{q^3[n-1]_{p,q}(p^2x+1-x)_{p,q}^{n-2}x^2}{[n]_{p,q}^3}+\frac{(3p+2pq+q^2)q^3[n-1]_{p,q}[n-2]_{p,q}(px+1-x)_{p,q}^{n-3}x^3}{[n]_{p,q}^3}$
$+\frac{q^6[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}x^4}{[n]_{p,q}^3}.$

Proof.

$$B_{n,p,q}(t^4;x)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k \prod_{s=0}^{n-k-1}(p^s-q^sx)\frac{[k]_{p,q}^4}{[n]_{p,q}^4}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} x^{k+1} \prod_{s=0}^{n-k-2} (p^s - q^s x) \frac{[k+1]_{p,q}^3}{[n]_{p,q}^3} \\
&= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} x^{k+1} \prod_{s=0}^{n-k-2} (p^s - q^s x) \frac{\left(p^k + q[k]_{p,q}\right)^3}{[n]_{p,q}^3} \\
&= \frac{(p^3 x + 1 - x)_{p,q}^{n-1} x}{[n]_{p,q}^3} + \frac{3p^2 q[n-1]_{p,q} (px + 1 - x)_{p,q}^{n-2} x^2}{[n]_{p,q}^3} \\
&\quad + \frac{3pq^2[n-1]_{p,q} (p^2 x + 1 - x)_{p,q}^{n-2} x^2}{[n]_{p,q}^3} + \frac{q^3[n-1]_{p,q} (p^2 x + 1 - x)_{p,q}^{n-2} x^2}{[n]_{p,q}^3} \\
&\quad + \frac{(3p + 2pq + q^2) q^3 [n-1]_{p,q} [n-2]_{p,q} (px + 1 - x)_{p,q}^{n-3} x^3}{[n]_{p,q}^3} \\
&\quad + \frac{q^6 [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q} x^4}{[n]_{p,q}^3}.
\end{aligned}
$$

□

Lemma 6.2. *For $x \in [0,1]$, $0 < q < p \leq 1$, we have*

(i) $B_{n,p,q}\big((t-x); x\big) = 0$;

(ii) $B_{n,p,q}\big((t-x)^2; x\big) = \frac{(px+1-x)_{p,q}^{n-1} x}{[n]_{p,q}} + \left(\frac{q[n-1]_{p,q}}{[n]_{p,q}} - 1\right) x^2$;

(iii)
$$
\begin{aligned}
B_{n,p,q}\big((t-x)^4; x\big) &= \frac{(p^3x+1-x)_{p,q}^{n-1}}{[n]_{p,q}^3} x \\
&+ \left\{ \frac{3p^2 q[n-1]_{p,q}(px+1-x)_{p,q}^{n-2}}{[n]_{p,q}^3} + \frac{(3p+q)q^2[n-1]_{p,q}(p^2x+1-x)_{p,q}^{n-2}}{[n]_{p,q}^3} - \frac{4(p^2x+1-x)_{p,q}^{n-1}}{[n]_{p,q}^2} \right\} x^2 \\
&+ \left\{ \frac{(3p+2pq+q^2)q^3[n-1]_{p,q}[n-2]_{p,q}(px+1-x)_{p,q}^{n-3}}{[n]_{p,q}^3} - \frac{4(2p+q)q[n-1]_{p,q}(px+1-x)_{p,q}^{n-2}}{[n]_{p,q}^2} + \frac{6(px+1-x)_{p,q}^{n-1}}{[n]_{p,q}} \right\} x^3 \\
&+ \left\{ \frac{q^6[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}{[n]_{p,q}^3} - \frac{4q^3[n-1]_{p,q}[n-2]_{p,q}}{[n]_{p,q}^2} + \frac{6[n-1]_{p,q}}{[n]_{p,q}} - 3 \right\} x^4.
\end{aligned}
$$

Proof. (ii) By linearity of the operators, we have

$$
\begin{aligned}
B_{n,p,q}\big((t-x)^2; x\big) &= B_{n,p,q}(t^2; x) - 2x B_{n,p,q}(t; x) + x^2 B_{n,p,q}(1; x) \\
&= \frac{x(px+1-x)_{p,q}^{n-1}}{[n]_{p,q}} + \frac{q[n-1]_{p,q}}{[n]_{p,q}} x^2 - 2x^2 + x^2 \\
&= \frac{x(px+1-x)_{p,q}^{n-1}}{[n]_{p,q}} + \left(\frac{q[n-1]_{p,q}}{[n]_{p,q}} - 1\right) x^2.
\end{aligned}
$$

(iii) Using Lemma 6.1 and by linearity of the operators, we can find our desired result. □

Remark 6.3. *For $q \in (0,1)$ and $p \in (q,1]$ it is obvious that $\lim_{n\to\infty} [n]_{p,q} = 0$ or $\frac{1}{p-q}$. In order to reach to convergence results of the operator $B_{n,p,q}$ we take sequences $q_n \in (0,1)$ and $p_n \in (q_n, 1]$ such that $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$. So we get that $\lim_{n\to\infty} [n]_{p_n,q_n} = \infty$.*

Thus the above remark provides an example that such a sequence can always be constructed. If we choose for $a > b > 0$, $q_n = \frac{n}{n+a} < \frac{n}{n+b} = p_n$ such that $0 < q_n < p_n \leq 1$, it can be easily seen that $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} p_n^n = e^{-b}$, $\lim_{n\to\infty} q_n^n = e^{-a}$. Hence we guarantee that $\lim_{n\to\infty} [n]_{p_n,q_n} = \infty$.

6.2.2 Approximation results

Theorem 6.4. *Let $0 < q_n < p_n \leq 1$ such that $\lim_{n\to\infty} p_n = 1$ and $\lim_{n\to\infty} q_n = 1$. Then for each $f \in C[0,1]$, $B_{n,p_n,q_n}(f;x)$ converges uniformly to f on $[0,1]$.*

Proof. By the Korovkin's theorem it suffices to show that

$$\lim_{n\to\infty} \|B_{n,p_n,q_n}(t^m;x) - x^m\|_{C[0,1]} = 0, \quad m = 0,1,2.$$

By Lemma 6.1(i)-(ii), it is clear that

$$\lim_{n\to\infty} \|B_{n,p_n,q_n}(1;x) - 1\|_{C[0,1]} = 0;$$

$$\lim_{n\to\infty} \|B_{n,p_n,q_n}(t;x) - x\|_{C[0,1]} = 0.$$

Using $q_n[n-1]_{p_n,q_n} = [n]_{p_n,q_n} - p_n^{n-1}$ and by Lemma 6.1(iii), we have

$$\begin{aligned} |B_{n,p_n,q_n}(t^2;x) - x^2|_{C[0,1]} &= \left| \frac{(xp_n + 1 - x)_{p_n,q_n}^{n-1} x}{[n]_{p_n,q_n}} + \left(\frac{q_n[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}} - 1 \right) x^2 \right| \\ &\leq \frac{(xp_n + 1 - x)_{p_n,q_n}^{n-1} x}{[n]_{p_n,q_n}} + \frac{p_n^{n-1}}{[n]_{p_n,q_n}} x^2. \end{aligned}$$

Taking maximum of both sides of the above inequality, we get

$$\|B_{n,p_n,q_n}(t^2;x) - x^2\|_{C[0,1]} \leq \frac{(1+p_n)_{p_n,q_n}^{n-1} + p_n^{n-1}}{[n]_{p_n,q_n}}$$

which yields

$$\lim_{n\to\infty} \|B_{n,p_n,q_n}(t^2;x) - x^2\|_{C[0,1]} = 0.$$

Thus the proof is completed. □

Now we calculate the order of convergence of the $B_{n,p,q}$ operators.

Theorem 6.5. *If $f \in C[0,1]$, then*

$$|B_{n,p,q}(f;x) - f(x)| \leq 2\omega\left(f, \sqrt{\frac{(1+p)_{p_n,q_n}^{n-1}}{[n]_{p,q}}}\right)$$

holds.

Proof. Since $B_{n,p,q}(1,x)=1$, we have

$$\left|B_{n,p,q}(f;x)-f(x)\right|\leq \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k \prod_{s=0}^{n-k-1}(p^s-q^sx)\left|f\left(\frac{[k]_{p,q}}{[n]_{p,q}}\right)-f(x)\right|.$$

In view of Lemma 6.2(ii), we get

$$\begin{aligned}
&\left|B_{n,p,q}(f;x)-f(x)\right| \\
&\leq \left\{\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k \prod_{s=0}^{n-k-1}(p^s-q^sx)\left(\frac{\left|\frac{[k]_{p,q}}{[n]_{p,q}}-x\right|^2}{\delta^2}+1\right)\right\}\omega(f,\delta) \\
&=\left\{\frac{1}{\delta^2}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k \prod_{s=0}^{n-k-1}(p^s-q^sx)\left(\frac{[k]_{p,q}}{[n]_{p,q}}-x\right)^2+\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k \prod_{s=0}^{n-k-1}(p^s-q^sx)\right\}\omega(f \\
&=\left\{\frac{1}{\delta^2}\left(B_{n,p,q}(t^2;x)-2xB_{n,p,q}(t;x)+x^2B_{n,p,q}(1;x)\right)+1\right\}\omega(f,\delta) \\
&=\left\{\frac{1}{\delta^2}\left(\frac{x(xp+1-x)_{p,q}^{n-1}}{[n]_{p,q}}+\left(\frac{q[n-1]_{p,q}}{[n]_{p,q}}-1\right)x^2\right)+1\right\}\omega(f,\delta) \\
&=\left\{\frac{1}{\delta^2}\left(\frac{x(xp+1-x)_{p,q}^{n-1}}{[n]_{p,q}}-\frac{p^{n-1}}{[n]_{p,q}}x^2\right)+1\right\}\omega(f,\delta) \\
&\leq \left\{\frac{1}{\delta^2}\left(\frac{x(xp+1-x)_{p,q}^{n-1}}{[n]_{p,q}}\right)+1\right\}w_f(\delta)\leq \left\{\frac{1}{\delta^2}\left(\frac{(1+p)_{p,q}^{n-1}}{[n]_{p,q}}\right)+1\right\}\omega(f,\delta).
\end{aligned}$$

Choosing $\delta=\delta_n=\sqrt{\frac{(1+p)_{p,q}^{n-1}}{[n]_{p,q}}}$, we have

$$\left|B_{n,p,q}(f;x)-f(x)\right|\leq 2\omega(f,\delta_n).$$

This completes the proof of the theorem. □

Now we give the rate of convergence of the operators $B_{n,p,q}$ in terms of the elements of the usual Lipschitz class $Lip_M(\alpha)$ (see section 4.2.2).

Theorem 6.6. *Let $0<q<p\leq 1$. Then for each $f\in Lip_M(\alpha)$ we have*

$$|B_{n,p,q}(f;x)-f(x)|\leq M\delta_n^{\alpha}(x),$$

where

$$\delta_n^2(x)=\left(\frac{q[n-1]_{p,q}}{[n]_{p,q}}-1\right)x^2+\frac{(xp+1-x)_{p,q}^{n-1}}{[n]_{p,q}}x.$$

Proof. Let us denote $P_{n,k}(x) = \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k \prod_{s=0}^{n-k-1}(p^s - q^s x)$. Then by the monotonicity of the operators $B_{n,p,q}$, we can write

$$\begin{aligned} |B_{n,p,q}(f;x) - f(x)| &\leq B_{n,p,q}\big(|f(t) - f(x)|;x\big) \\ &\leq \sum_{k=0}^{n} P_{n,k}(x)\left|f\left(\frac{[k]_{p,q}}{[n]_{p,q}}\right) - f(x)\right| \leq M\sum_{k=0}^{n} P_{n,k}(x)\left|\frac{[k]_{p,q}}{[n]_{p,q}} - x\right|^{\alpha} \\ &= M\sum_{k=0}^{n}\left(P_{n,k}(x)\left(\frac{[k]_{p,q}}{[n]_{p,q}} - x\right)^2\right)^{\alpha/2} P_{n,k}^{\frac{2-\alpha}{2}}(x). \end{aligned}$$

Now applying the Hölder's inequality for the sum and taking into consideration Lemma 6.1(i) and Lemma 6.2(ii), we have

$$\begin{aligned} &|B_{n,p,q}(f;x) - f(x)| \\ &\leq M\left(\sum_{k=0}^{n} P_{n,k}(x)\left(\frac{[k]_{p,q}}{[n]_{p,q}} - x\right)^2\right)^{\alpha/2}\left(\sum_{k=0}^{n} P_{n,k}(x)\right)^{\frac{2-\alpha}{2}} = M\left\{B_{n,p,q}\big((t-x)^2;x\big)\right\}^{\frac{\alpha}{2}}. \end{aligned}$$

Choosing $\delta : \delta_n(x) = \sqrt{B_{n,p,q}\big((t-x)^2;x\big)}$, we obtain

$$|B_{n,p,q}(f;x) - f(x)| \leq M\delta_n^{\alpha}(x).$$

Hence, the desired result is obtained. □

6.3 (p,q)-Bernstein-Stancu opertaors

Now, we introduce (p,q)-analogue of Bernstein-Stancu operators as

$$S_{n,p,q}(f;x) = \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k \prod_{s=0}^{n-k-1}(p^s - q^s x)\ f\left(\frac{[k]_{p,q} + \alpha}{[n]_{p,q} + \beta}\right), \quad x \in [0,1]. \tag{6.3}$$

6.3.1 Auxiliary results

Lemma 6.7. *For* $x \in [0,1]$, $0 < q < p \leq 1$

(i) $S_{n,p,q}(1;x) = 1$;

(ii) $S_{n,p,q}(t;x) = \frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha}{[n]_{p,q}+\beta}$;

(iii) $S_{n,p,q}(t^2;x) = \frac{1}{[n]_{p,q}+\beta^2}\left[[n]_{p,q}(xp+1-x)_{p,q}^{n-1}x+q[n]_{p,q}[n-1]_{p,q}x^2+2\alpha[n]_{p,q}x+\alpha^2\right].$

Lemma 6.8. *For* $x \in [0,1]$, $0 < q < p \leq 1$

(i) $S_{n,p,q}\big((t-x);x\big) = -\frac{\beta x-\alpha}{[n]_{p,q}+\beta}$;

(ii) $S_{n,p,q}\big((t-x)^2;x\big) = \frac{[n]_{p,q}(px+1-x)_{p,q}^{n-1}x}{\left([n]_{p,q}+\beta\right)^2} + \frac{\beta^2-p^{n-1}[n]_{p,q}}{\left([n]_{p,q}+\beta\right)^2} - \frac{2\alpha\beta}{\left([n]_{p,q}+\beta\right)^2}x + \frac{\alpha^2}{\left([n]_{p,q}+\beta\right)^2}.$

Lemma 6.9. *For sequences* (p_n), (q_n) *as* $0 < q_n < p_n \leq 1$ *such that* $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$, *and satisfying* $\lim_{n\to\infty} (p_n)^n \to a\ (0 \leq a \leq 1)$, $\lim_{n\to\infty} (q_n)^n \to b\ (0 \leq b < 1)$, *we have*

(i) $\lim_{n\to\infty} [n]_{p_n,q_n} S_{n,p_n,q_n}\big((t-x);x\big) = -\beta x + \alpha$;

(ii) $\lim_{n\to\infty} [n]_{p_n,q_n} S_{n,p_n,q_n}\big((t-x)^2;x\big) = -ax^2 + 2\beta x(1-x)$.

Proof. (i)

$$\lim_{n\to\infty} [n]_{p_n,q_n} S_{n,p_n,q_n}\big((t-x);x\big) = \lim_{n\to\infty} [n]_{p_n,q_n} \frac{\alpha-\beta x}{[n]_{p_n,q_n}+\beta} = -\beta x + \alpha.$$

(ii)

$$\lim_{n\to\infty} [n]_{p_n,q_n} S_{n,p_n,q_n}\big((t-x)^2;x\big)$$
$$= \lim_{n\to\infty} [n]_{p_n,q_n} \left\{ S_{n,p_n,q_n}\big(t^2;x\big) - x^2 - 2x S_{n,p_n,q_n}\big((t-x);x\big) \right\}$$
$$= \lim_{n\to\infty} [n]_{p_n,q_n} \left\{ \frac{[n]_{p_n,q_n}(xp_n+1-x)_{p_n,q_n}^{n-1}x + q_n[n]_{p_n,q_n}[n-1]_{p_n,q_n}x^2 + 2\alpha[n]_{p_n,q_n}x + \alpha^2}{\left([n]_{p_n,q_n}+\beta\right)^2} \right\}$$
$$- \lim_{n\to\infty} [n]_{p_n,q_n} x^2 - 2x \lim_{n\to\infty} [n]_{p_n,q_n} S_{n,p_n,q_n}\big((t-x);x\big)$$
$$= \lim_{n\to\infty} [n]_{p_n,q_n}$$
$$\left\{ \frac{[n]_{p_n,q_n}(xp_n+1-x)_{p_n,q_n}^{n-1}x + [n]_{p_n,q_n}\big([n]_{p_n,q_n} - p^{n-1}\big)x^2 + 2\alpha[n]_{p_n,q_n}x + \alpha^2 - \left([n]_{p_n,q_n}+\beta\right)^2 x}{\left([n]_{p_n,q_n}+\beta\right)^2} \right.$$
$$- 2x \lim_{n\to\infty} [n]_{p_n,q_n} S_{n,p_n,q_n}\big((t-x);x\big)$$
$$= \lim_{n\to\infty} [n]_{p_n,q_n}$$
$$\left\{ \frac{[n]_{p_n,q_n}(xp_n+1-x)_{p_n,q_n}^{n-1}x - p_n^{n-1}[n]_{p_n,q_n}x^2 + 2\alpha[n]_{p_n,q_n}x + \alpha^2 - \beta^2x^2 - 2\beta[n]_{p_n,q_n}x}{\left([n]_{p_n,q_n}+\beta\right)^2} \right\}$$
$$- 2x \lim_{n\to\infty} [n]_{p_n,q_n} S_{n,p_n,q_n}\big((t-x);x\big)$$

$$= \lim_{n\to\infty} (xp_n + 1 - x)^{n-1}_{p_n,q_n} x - \lim_{n\to\infty} p_n^{n-1} [n]_{p_n,q_n} x^2 + 2\alpha x - 2\beta x - 2x(-\beta x + \alpha)$$
$$= -ax^2 + 2\beta x(1-x).$$

$\square$

6.3.2 Approximation results

Theorem 6.10. *Let* $0 < q_n < p_n \leq 1$ *such that* $\lim_{n\to\infty} p_n = 1$ *and* $\lim_{n\to\infty} q_n = 1$. *Then for each* $f \in C[0,1]$, $S_{n,p_n,q_n}(f;x)$ *converges uniformly to* f *on* $[0,1]$.

Proof. see Theorem 3.1 in [79]. $\square$

Theorem 6.11. *Let* $f \in C[0,1]$ *and* $0 < q < p \leq 1$. *Then for all* $n \in \mathbb{N}$, *there exists an absolute constant* $C > 0$ *such that*

$$\left|S_{n,p,q}(f;x) - f(x)\right| \leq C\omega_2\big(f, \delta_n(x)\big) + \omega\left(f, \frac{|\alpha - \beta x|}{[n]_{p,q} + \beta}\right),$$

where

$$\delta_n(x) = \sqrt{S_{n,p,q}\left((t-x)^2; x\right) + \left(\frac{\alpha - \beta x}{[n]_{p,q} + \beta}\right)^2}.$$

Proof. For $x \in [0,1]$, we consider the auxiliary operators $S^*_{n,p,q}$ defined by

$$S^*_{n,p,q}(f;x) = S_{n,p,q}(f;x) + f(x) - f\left(\frac{[n]_{p,q}x + \alpha}{[n]_{p,q} + \beta}\right).$$

From Lemma 6.7, we observe that the operators $S^*_{n,p,q}(f;x)$ are linear and reproduce the linear functions. Hence

$$\begin{aligned} S^*_{n,p,q}(t-x;x) &= S_{n,p,q}(t-x;x) - \left(\frac{[n]_{p,q}x + \alpha}{[n]_{p,q} + \beta} - x\right) \\ &= S_{n,p,q}(t;x) - xS_{n,p,q}(1;x) - \frac{[n]_{p,q}x + \alpha}{[n]_{p,q} + \beta} + x = 0. \end{aligned}$$

Let $x \in [0,1]$ and $g \in C^2[0,1]$. Using the Taylor's formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)\, g''(u)\, du.$$

Applying $S^*_{n,p,q}$ to both sides of the above equation, we have

$$S^*_{n,p,q}(g;x) - g(x) = S^*_{n,p,q}\big((t-x)g'(x);x\big) + S^*_{n,p,q}\left(\int_x^t (t-u)g''(u)du; x\right)$$

$$
\begin{aligned}
&= g'(x)S^*_{n,p,q}\big((t-x);x\big) + S_{n,p,q}\bigg(\int_x^t (t-u)g''(u)du;x\bigg)\\
&\quad -\int_x^{\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}} \bigg(\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}-u\bigg)g''(u)du\\
&= S_{n,p,q}\bigg(\int_x^t (t-u)g''(u)du;x\bigg)\\
&\quad -\int_x^{\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}} \bigg(\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}-u\bigg)g''(u)du.
\end{aligned}
$$

On the other hand, since

$$
\bigg|\int_x^t (t-u)g''(u)du\bigg| \le \int_x^t |t-u||g''(u)|du \le \|g''\|\int_x^t |t-u|du \le (t-x)^2\|g''\|
$$

and

$$
\bigg|\int_x^{\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}} \bigg(\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}-u\bigg)g''(u)du\bigg| \le \bigg(\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}-x\bigg)^2\|g''\| = \bigg(\frac{\alpha-\beta x}{[n]_{p,q}+\beta}\bigg)^2\|g''\|
$$

we conclude that

$$
\begin{aligned}
\big|S^*_{n,p,q}(g;x)-g(x)\big| &= \bigg|S_{n,p,q}\bigg(\int_x^t (t-u)g''(u)du;x\bigg)\\
&\quad -\int_x^{\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}} \bigg(\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}-u\bigg)g''(u)du\bigg|\\
&\le \|g''\|S_{n,p,q}\left((t-x)^2;x\right) + \bigg(\frac{\alpha-\beta x}{[n]_{p,q}+\beta}\bigg)^2\|g''\|\\
&= \delta_n^2(x)\|g''\|.
\end{aligned}
$$

Now, taking into account boundedness of $S^*_{n,p,q}$, we have

$$
\big|S^*_{n,p,q}(f;x)\big| \le \big|S_{n,p,q}(f;x)\big| + 2\|f\| \le 3\|f\|.
$$

Therefore

$$
\begin{aligned}
\big|S_{n,p,q}(f;x)-f(x)\big| &\le \big|S^*_{n,p,q}(f;x)-f(x)\big| + \bigg|f(x)-f\bigg(\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}\bigg)\bigg|\\
&\le \big|S^*_{n,p,q}(f-g;x)-(f-g)(x)\big|\\
&\quad +\bigg|f(x)-f\bigg(\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}\bigg)\bigg| + \big|S^*_{n,p,q}(g;x)-g(x)\big|\\
&\le \big|S^*_{n,p,q}(f-g;x)\big| + \big|(f-g)(x)\big|
\end{aligned}
$$

$$+\left|f(x)-f\left(\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}\right)\right|+\left|S^*_{n,p,q}(g;x)-g(x)\right|$$
$$\leq \;4\|f-g\|+\omega\left(f;\frac{|\alpha-\beta x|}{[n]_{p,q}+\beta}\right)+\delta_n^2(x)\|g''\|.$$

Hence, taking the infimum on the right-hand side over all $g \in C^2[0,1]$, we have the following result

$$\left|S_{n,p,q}(f;x)-f(x)\right| \leq 4K_2\left(f,\delta_n^2(x)\right)+\omega\left(f;\frac{|\alpha-\beta x|}{[n]_{p,q}+\beta}\right).$$

In view of the property of K-functional, we get

$$\left|S_{n,p,q}(f;x)-f(x)\right| \leq C\omega_2\left(f,\delta_n(x)\right)+\omega\left(f;\frac{|\alpha-\beta x|}{[n]_{p,q}+\beta}\right).$$

This completes the proof of the theorem. □

Let $C[0,1]$ be the space of all continuous functions.

Theorem 6.12. *Let $f \in C[0,1]$ be such that $f', f'' \in C[0,1]$, and the sequences (p_n), (q_n) as $0 < q_n < p_n \leq 1$ such that $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$, and satisfying $\lim_{n\to\infty}(p_n)^n \to a$, $\lim_{n\to\infty}(q_n)^n \to b$. Then the following equality holds*

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(S_{n,p_n,q_n}(f;x)-f(x)\right)=(-\beta x+\alpha)f'(x)+\frac{\left(-ax^2+2\beta x(1-x)\right)}{2}f''(x)$$

uniformly on $[0,1]$.

Proof. By the Taylor's formula we may write

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2 \tag{6.4}$$

where $r(t,x)$ is the Peano's form of remainder and $\lim_{t\to x} r(t,x)=0$. Applying $S_{n,p_n,q_n}(f;x)$ to (6.4), we obtain

$$\begin{aligned}
&[n]_{p_n,q_n}\left(S_{n,p_n,q_n}(f;x)-f(x)\right)\\
&=[n]_{p_n,q_n}S_{n,p_n,q_n}\left((t-x);x\right)f'(x)\\
&+[n]_{p_n,q_n}S_{n,p_n,q_n}\left((t-x)^2;x\right)\frac{f''(x)}{2}+[n]_{p_n,q_n}S_{n,p_n,q_n}\left(r(t,x)(t-x)^2;x\right).
\end{aligned}$$

By the Cauchy-Schwartz inequality, we have

$$S_{n,p_n,q_n}\left(r(t,x)(t-x)^2;x\right) \leq \sqrt{S_{n,p_n,q_n}\left(r^2(t,x);x\right)} \times \sqrt{S_{n,p_n,q_n}\left(r(t,x)(t-x)^4;x\right)} \tag{6.5}$$

Observe that $r^2(x,x)=0$ and $r^2(.,x)\in C[0,1]$, then it follows from Theorem 6.10 , that

$$S_{n,p_n,q_n}\left(r^2(t,x);x\right)=r^2(x,x)=0 \tag{6.6}$$

uniformly with respect to $x\in[0,1]$. Now from (6.5), (6.6) and Lemma 6.9(ii), we get

$$\lim_{n\to\infty}[n]_{p_n,q_n}S_{n,p_n,q_n}\left(r(t,x)(t-x)^2;q;x\right)=0.$$

Finally using Lemma 6.9 , we get the following

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{p_n,q_n}\left(S_{n,p_n,q_n}(f;x)-f(x)\right)\\
&=\lim_{n\to\infty}[n]_{p_n,q_n}S_{n,p_n,q_n}\left((t-x);x\right)f'(x)+\lim_{n\to\infty}[n]_{p_n,q_n}S_{n,p_n,q_n}\left((t-x)^2;x\right)\frac{f''(x)}{2}\\
&+\lim_{n\to\infty}[n]_{p_n,q_n}S_{n,p_n,q_n}\left(r(t,x)(t-x)^2;x\right)=(-\beta x+\alpha)f'(x)+\frac{\left(-\alpha x^2+2\beta x(1-x)\right)}{2}f''(x).
\end{aligned}$$

This completes the proof of the theorem. □

Chapter 7

On Kantorovich variant of (p, q)-Bernstein and (p, q)-Szász operators

7.1 Introduction and preliminaries

The (p, q)-integer was introduced in order to generalize or unify several forms of q-oscillator algebras well known in the earlier physics literature related to the representation theory of single parameter quantum algebras [18].

The aim of this chapter is to introduce a new generalization of q-Bernstein-Kantorovich operators and Szász-Kantorovich operators by using the notion of (p, q)-calculus and we call it as (p, q)-Bernstein-Kantorovich operators and (p, q)-Szász-Kantorovich operators, respectively. We study the approximation properties based on Korovkin's type approximation theorem and also establish some direct theorems for these operators. Further, we find the order of convergence and prove Voronovskaja type asymptotic formula.

7.2 (p, q)-Bernstein-Kantorovich operators

Dalmanoglu [21] defined the Bernstein-Kantorovich [56] operators using q-calculus as follows:

$$K_{n,q}(f;x) = [n+1]_q \sum_{k=0}^{n} p_{n,k}(q;x) \int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} f(t) d_q t, \quad x \in [0,1], \tag{7.1}$$

$$p_{n,k}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x),$$

where $K_{n,q} : C[0,1] \to C[0,1]$ are defined for any $n \in \mathbb{N}$ and for any function $f \in C[0,1]$.

Now, we introduce (p,q)-analogue of Bernstein-Kantorovich operators as

$$K_n^{(p,q)}(f;x)=\sum_{k=0}^{n} b_{n,k}^{(p,q)}(x)\frac{[n+1]_{p,q}}{[k+1]_{p,q}-[k]_{p,q}}\int_{\frac{[k]_{p,q}}{[n+1]_{p,q}}}^{\frac{[k+1]_{p,q}}{[n+1]_{p,q}}} f(t)d_{p,q}t,\;\; x\in[0,1], \tag{7.2}$$

where

$$b_{n,k}^{(p,q)}(x)=\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k(1-x)_{p,q}^{n-k}=\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k\prod_{s=0}^{n-k-1}(p^s-q^s x).$$

For $p=1$, operators (7.2) turns out to be the classical q-Bernstein-Kantorovich operators (7.1).

First, we prove the following basic lemmas:

Lemma 7.1. *For $x\in[0,1]$, $0<q<p\leq 1$*

(i) $K_n^{(p,q)}(1;x)=1;$

(ii) $K_n^{(p,q)}(t;x)=\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}}+\frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}};$

(iii) $K_n^{(p,q)}(t^2;x)=\frac{(q^2x+1-x)_{p,q}^n}{[3]_{p,q}[n+1]_{p,q}^2}+\frac{(p^2+2pq+p+q+1)(qx+1-x)_{p,q}^{n-1}[n]_{p,q}x}{[3]_{p,q}[n+1]_{p,q}^2}+\frac{p(p^2+p+1)[n]_{p,q}[n-1]_{p,q}x^2}{[3]_{p,q}[n+1]_{p,q}^2};$

(iv)
$$K_n^{(p,q)}((t-x)^2;x)=\frac{(q^2x+1-x)_{p,q}^n}{[3]_{p,q}[n+1]_{p,q}^2}+\left(\frac{(p^2+2pq+p+q+1)(qx+1-x)_{p,q}^{n-1}[n]_{p,q}}{[3]_{p,q}[n+1]_{p,q}^2}-\frac{2(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}}\right)x$$
$$+\left(\frac{p(p^2+p+1)[n]_{p,q}[n-1]_{p,q}}{[3]_{p,q}[n+1]_{p,q}^2}-\frac{2(p+1)[n]_{p,q}}{[2]_{p,q}[n+1]_{p,q}}+1\right)x^2.$$

Proof. (i)

$$K_n^{(p,q)}(1;x)=\sum_{k=0}^{n} b_{n,k}^{(p,q)}(x)\frac{[n+1]_{p,q}}{[k+1]_{p,q}-[k]_{p,q}}\int_{\frac{[k]_{p,q}}{[n+1]_{p,q}}}^{\frac{[k+1]_{p,q}}{[n+1]_{p,q}}} d_{p,q}t=1.$$

(ii)

$$K_n^{(p,q)}(t;x)=\sum_{k=0}^{n} b_{n,k}^{(p,q)}(x)\frac{[n+1]_{p,q}}{[k+1]_{p,q}-[k]_{p,q}}\int_{\frac{[k]_{p,q}}{[n+1]_{p,q}}}^{\frac{[k+1]_{p,q}}{[n+1]_{p,q}}} t\; d_{p,q}t$$
$$=\frac{1}{[2]_{p,q}[n+1]_{p,q}}\sum_{k=0}^{n} b_{n,k}^{(p,q)}(x)\big([k+1]_{p,q}+[k]_{p,q}\big).$$

Using $[k+1]_{p,q}=q^k+p[k]_{p,q}$, we have

$K_n^{(p,q)}(t;x)$

$$=\frac{1}{[2]_{p,q}[n+1]_{p,q}}\left\{\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}(qx)^k(1-x)_{p,q}^{n-k}+(p+1)\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}x^k(1-x)_{p,q}^{n-k}[k]_{p,q}\right\}$$

$$= \frac{1}{[2]_{p,q}[n+1]_{p,q}}\left\{(qx+1-x)^n_{p,q}+(p+1)[n]_{p,q}\sum_{k=0}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^{k+1}(1-x)^{n-k-1}_{p,q}\right\}$$
$$= \frac{(qx+1-x)^n_{p,q}}{[2]_{p,q}[n+1]_{p,q}}+\frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}}.$$

(iii)

$$K_n^{(p,q)}(t^2;x)$$
$$= \sum_{k=0}^{n} b_{n,k}^{(p,q)}(x)\frac{[n+1]_{p,q}}{[k+1]_{p,q}-[k]_{p,q}}\int_{\frac{[k]_{p,q}}{[n+1]_{p,q}}}^{\frac{[k+1]_{p,q}}{[n+1]_{p,q}}} t^2\, d_{p,q}t$$
$$= \frac{1}{[3]_{p,q}[n+1]^2_{p,q}}\sum_{k=0}^{n} b_{n,k}^{(p,q)}(x)\left([k+1]^2_{p,q}+[k]_{p,q}[k+1]_{p,q}+[k]^2_{p,q}\right)$$
$$= \frac{1}{[3]_{p,q}[n+1]^2_{p,q}}\sum_{k=0}^{n} b_{n,k}^{(p,q)}(x)\left((q^k+p[k]_{p,q})^2+[k]_{p,q}(q^k+p[k]_{p,q})+[k]^2_{p,q}\right)$$
$$= \frac{1}{[3]_{p,q}[n+1]^2_{p,q}}\left\{\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k(1-x)^{n-k}_{p,q}q^{2k}+(2p+1)\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k(1-x)^{n-k}_{p,q}q^k[k]_{p,q}\right.$$
$$\left.+(p^2+p+1)\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k(1-x)^{n-k}_{p,q}[k]^2_{p,q}\right\}$$
$$= \frac{1}{[3]_{p,q}[n+1]^2_{p,q}}\left\{(q^2x+1-x)^n_{p,q}+(2p+1)[n]_{p,q}\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} x^{k+1}(1-x)^{n-k-1}_{p,q}q^{k+1}\right.$$
$$\left.+(p^2+p+1)[n]_{p,q}\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} x^{k+1}(1-x)^{n-k-1}_{p,q}[k+1]_{p,q}\right\}$$
$$= \frac{1}{[3]_{p,q}[n+1]^2_{p,q}}\left\{(q^2x+1-x)^n_{p,q}+(2p+1)q(qx+1-x)^{n-1}[n]_{p,q}x\right.$$
$$+(p^2+p+1)(qx+1-x)^{n-1}_{p,q}[n]_{p,q}x$$
$$\left.+p(p^2+p+1)[n]_{p,q}[n-1]_{p,q}\sum_{k=0}^{n-2}\begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} x^{k+2}(1-x)^{n-k-2}_{p,q}\right\}$$
$$= \frac{(q^2x+1-x)^n_{p,q}}{[3]_{p,q}[n+1]^2_{p,q}}+\frac{(p^2+2pq+p+q+1)(qx+1-x)^{n-1}_{p,q}[n]_{p,q}x}{[3]_{p,q}[n+1]^2_{p,q}}$$
$$+\frac{p(p^2+p+1)[n]_{p,q}[n-1]_{p,q}x^2}{[3]_{p,q}[n+1]^2_{p,q}}.$$

(iv) Using the linearity of the operators $K_n^{(p,q)}$ and (i)-(iii), we have our desired result. $\square$

7.2.1 Korovkin type approximation

Theorem 7.2. *Let $0 < q_n < p_n \leq 1$ such that $\lim\limits_{n\to+\infty} p_n = 1$ and $\lim\limits_{n\to+\infty} q_n = 1$. Then for each $f \in C[0,1]$, $K_n^{(p_n,q_n)}(f;x)$ converges uniformly to f on $[0,1]$.*

Proof. Proof is same as of Theorem 6.4. □

Now we will compute the rate of convergence in terms of modulus of continuity.

7.2.2 Order of approximation

Let $f \in C[0,1]$. The modulus of continuity of f denoted by $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by the relation (1.3).

Theorem 7.3. *If $f \in C[0,1]$, then*

$$\left|K_n^{(p,q)}(f;x) - f(x)\right| \leq 2\omega(f,\delta_n)$$

takes place, where δ_n

$$\begin{aligned} &= \frac{1}{[n+1]_{p,q}}\Big\{\big([2]_{p,q}q^{2n} + [2]_{p,q}(p^2+2pq+p+q+1)q^{n-1}[n]_{p,q} \\ &\quad +[2]_{p,q}p(p^2+p+1)[n]_{p,q}[n-1]_{p,q} - 2[3]_{p,q}(p+1)[n]_{p,q}[n+1]_{p,q} \\ &\quad +[2]_{p,q}[3]_{p,q}[n+1]_{p,q}^2\big)/([2]_{p,q}[3]_{p,q})\Big\}^{1/2}. \end{aligned}$$

Proof. Since $K_n^{(p,q)}(1;x) = 1$, we have

$$\begin{aligned} &\left|K_n^{(p,q)}(f;x) - f(x)\right| \\ &\leq K_n^{(p,q)}\left(|f(t)-f(x)|;x\right) \\ &\leq \sum_{k=0}^{n} b_{n,k}^{(p,q)}(x)\frac{[n+1]_{p,q}}{[k+1]_{p,q}-[k]_{p,q}}\int_{\frac{[k]_{p,q}}{[n+1]_{p,q}}}^{\frac{[k+1]_{p,q}}{[n+1]_{p,q}}} |f(t)-f(x)|d_{p,q}t. \end{aligned}$$

In view of (1.6), we get

$$\begin{aligned} &\left|K_n^{(p,q)}(f;x) - f(x)\right| \\ &\leq \left\{\sum_{k=0}^{n} b_{n,k}^{(p,q)}(x)\frac{[n+1]_{p,q}}{[k+1]_{p,q}-[k]_{p,q}}\int_{\frac{[k]_{p,q}}{[n+1]_{p,q}}}^{\frac{[k+1]_{p,q}}{[n+1]_{p,q}}}\left(\frac{|t-x|^2}{\delta^2}+1\right)d_{p,q}t\right\}\omega(f,\delta) \end{aligned}$$

$$
\begin{aligned}
&= \left\{\frac{1}{\delta^2}K_n^{(p,q)}\left((t-x)^2;x\right)+1\right\}\omega(f,\delta)\\
&= \left\{\frac{1}{\delta^2}\left\{\frac{(q^2x+1-x)_{p,q}^n}{[3]_{p,q}[n+1]_{p,q}^2}\right.\right.\\
&+\left(\frac{(p^2+2pq+p+q+1)(qx+1-x)_{p,q}^{n-1}[n]_{p,q}}{[3]_{p,q}[n+1]_{p,q}^2}-\frac{2(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}}\right)x\\
&+\left.\left.\left(\frac{p(p^2+p+1)[n]_{p,q}[n-1]_{p,q}}{[3]_{p,q}[n+1]_{p,q}^2}-\frac{2(p+1)[n]_{p,q}}{[2]_{p,q}[n+1]_{p,q}}+1\right)x^2\right\}+1\right\}\omega(f,\delta)\\
&\leq \left\{\frac{1}{\delta^2}\left\{\frac{(q^2x+1-x)_{p,q}^n}{[3]_{p,q}[n+1]_{p,q}^2}\right.\right.\\
&+\left(\frac{(p^2+2pq+p+q+1)(qx+1-x)_{p,q}^{n-1}[n]_{p,q}x}{[3]_{p,q}[n+1]_{p,q}^2}\right)\\
&+\left.\left.\left(\frac{p(p^2+p+1)[n]_{p,q}[n-1]_{p,q}}{[3]_{p,q}[n+1]_{p,q}^2}-\frac{2(p+1)[n]_{p,q}}{[2]_{p,q}[n+1]_{p,q}}+1\right)x^2\right\}+1\right\}\omega(f,\delta)\\
&\leq \left\{\frac{1}{\delta^2}\left(\frac{q^{2n}}{[3]_{p,q}[n+1]_{p,q}^2}+\frac{(p^2+2pq+p+q+1)q^{n-1}[n]_{p,q}}{[3]_{p,q}[n+1]_{p,q}^2}\right.\right.\\
&+\left.\left.\frac{p(p^2+p+1)[n]_{p,q}[n-1]_{p,q}}{[3]_{p,q}[n+1]_{p,q}^2}-\frac{2(p+1)[n]_{p,q}}{[2]_{p,q}[n+1]_{p,q}}+1\right)+1\right\}\omega(f,\delta)\\
&= \left\{\frac{1}{\delta^2}\left\{\Big([2]_{p,q}q^{2n}+[2]_{p,q}(p^2+2pq+p+q+1)q^{n-1}[n]_{p,q}\right.\right.\\
&+[2]_{p,q}p(p^2+p+1)[n]_{p,q}[n-1]_{p,q}-2[3]_{p,q}(p+1)[n]_{p,q}[n+1]_{p,q}\\
&+\left.\left.[2]_{p,q}[3]_{p,q}[n+1]_{p,q}^2\Big)\Big/\Big([2]_{p,q}[3]_{p,q}[n+1]_{p,q}^2\Big)\right\}^{1/2}+1\right\}\omega(f,\delta).
\end{aligned}
$$

Choosing

$$
\begin{aligned}
\delta=\delta_n=\frac{1}{[n+1]_{p,q}}&\\
&\Big\{\big([2]_{p,q}q^{2n}+[2]_{p,q}(p^2+2pq+p+q+1)q^{n-1}[n]_{p,q}+[2]_{p,q}p(p^2+p+1)[n]_{p,q}[n-1]_{p,q}\\
&-2[3]_{p,q}(p+1)[n]_{p,q}[n+1]_{p,q}+[2]_{p,q}[3]_{p,q}[n+1]_{p,q}^2\big)/([2]_{p,q}[3]_{p,q})\Big\}^{1/2}
\end{aligned}
$$

we have

$$\left|K_n^{(p,q)}(f;x)-f(x)\right|\leq 2\omega(f,\delta_n).$$

This completes the proof of the theorem. □

Next we prove the local approximation property for the operators $K_n^{(p,q)}$.

7.2.3 Local approximation property

Theorem 7.4. *Let $f \in C[0,1]$ and $0 < q < p \leq 1$. Then for all $n \in \mathbb{N}$, there exists an absolute constant $C > 0$ such that*

$$\left|K_n^{(p,q)}(f;x) - f(x)\right| \leq C\omega_2\big(f, \delta_n(x)\big) + \omega\big(f, \alpha_n(x)\big),$$

where

$$\delta_n(x) = \left\{K_n^{(p,q)}\left((t-x)^2;x\right) + \left(\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}} + \frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}} - x\right)^2\right\}^{1/2}$$

and

$$\alpha_n(x) = \left|\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}} + \left(\frac{(p+1)[n]_{p,q}}{[2]_{p,q}[n+1]_{p,q}} - 1\right)x\right|$$

and ω and ω_2 are first and second order moduli of continuity given by the expression (1.3) and (1.4), respectively.

Proof. For $x \in [0,1]$, we consider the auxiliary operators K_n^* defined by

$$K_n^*(f;x) = K_n^{(p,q)}(f;x) + f(x) - f\left(\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}} + \frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}}\right).$$

From Lemma 7.1, we observe that the operators $K_n^*(f;x)$ are linear and reproduce the linear functions. Hence

$$\begin{aligned}
&K_n^*(t-x;x)\\
&= K_n^{(p,q)}(t-x;x) - \left(\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}} + \frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}} - x\right)\\
&= K_n^{(p,q)}(t;x) - xK_n^{(p,q)}(1;x) - \left(\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}} + \frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}}\right) + x = 0.
\end{aligned}$$

Let $x \in [0,1]$ and $g \in C_B^2[0,1]$. Using the Taylor's formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)\, g''(u)\, du.$$

Applying K_n^* to both sides of the above equation, we have

$$\begin{aligned}
&K_n^*(g;x) - g(x)\\
&= K_n^*\big((t-x)g'(x);x\big) + K_n^*\left(\int_x^t (t-u)g''(u)du;x\right)
\end{aligned}$$

$$= g'(x)K_n^*\big((t-x);x\big) + K_n^{(p,q)}\left(\int_x^t (t-u)g''(u)du;x\right)$$
$$-\int_x^{\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}}+\frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}}} \left(\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}} + \frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}} - u\right)g''(u)du$$
$$= K_n^{(p,q)}\left(\int_x^t (t-u)g''(u)du;x\right)$$
$$-\int_x^{\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}}+\frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}}} \left(\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}} + \frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}} - u\right)g''(u)du.$$

On the other hand, since

$$\left|\int_x^t (t-u)g''(u)du\right| \le \int_x^t |t-u||g''(u)|du \le \|g''\|\int_x^t |t-u|du \le (t-x)^2\|g''\|$$

and

$$\left|\int_x^{\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}}+\frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}}} \left(\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}} + \frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}} - u\right)g''(u)du\right|$$
$$\le \left(\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}} + \frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}} - x\right)^2\|g''\|$$

we conclude that

$$\left|K_n^*(g;x) - g(x)\right|$$
$$= \left|K_n^{(p,q)}\left(\int_x^t (t-u)g''(u)du;x\right)\right.$$
$$\left.-\int_x^{\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}}+\frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}}} \left(\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}} + \frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}} - u\right)g''(u)du\right|$$
$$\le \|g''\|K_n^{(p,q)}\left((t-x)^2;x\right) + \left(\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}} + \frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}} - x\right)^2\|g''\|$$
$$= \delta_n^2(x)\|g''\|.$$

Now, taking into account boundedness of K_n^*, we have

$$\left|K_n^*(f;x)\right| \le \left|K_n^{(p,q)}(f;x)\right| + 2\|f\| \le 3\|f\|.$$

Therefore

$$\left|K_n^{(p,q)}(f;x) - f(x)\right|$$

$$
\begin{aligned}
&\leq \left|K_n^*(f;x)-f(x)\right| + \left|f(x)-f\left(\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}}+\frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}}\right)\right| \\
&\leq \left|K_n^*(f-g;x)-(f-g)(x)\right| \\
&+\left|f(x)-f\left(\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}}+\frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}}\right)\right| + \left|K_n^*(g;x)-g(x)\right| \\
&\leq \left|K_n^*(f-g;x)\right| + \left|(f-g)(x)\right| \\
&+\left|f(x)-f\left(\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}}+\frac{(p+1)[n]_{p,q}x}{[2]_{p,q}[n+1]_{p,q}}\right)\right| + \left|K_n^*(g;x)-g(x)\right| \\
&\leq 4\|f-g\| + \omega\left(f;\left|\frac{(qx+1-x)_{p,q}^n}{[2]_{p,q}[n+1]_{p,q}}+\left(\frac{(p+1)[n]_{p,q}}{[2]_{p,q}[n+1]_{p,q}}-1\right)x\right|\right) + \delta_n^2(x)\|g''\|.
\end{aligned}
$$

Hence, taking the infimum on the right-hand side over all $g \in C^2[0,1]$ and using (1.8), we have the following result

$$\left|K_n^{(p,q)}(f;x)-f(x)\right| \leq 4K_2\big(f,\delta_n^2(x)\big)+\omega\big(f,\alpha_n(x)\big).$$

In view of the property of K-functional (1.9), we get

$$\left|K_n^{(p,q)}(f;x)-f(x)\right| \leq C\omega_2\big(f,\delta_n(x)\big)+\omega\big(f,\alpha_n(x)\big).$$

This completes the proof of the theorem. □

7.3 (p,q)-Szász-Kantorovich operators

Aral [4] and Aral and Gupta [5] proposed and studied some q analogue of Szász-Mirakjan operators [104], defined on positive real axis. In [70], Mahmudov introduced a q-parametric Szász-Mirakjan operators and studied their convergence properties. Recently, Acar [1] constructed a new operator with the help of (p,q)-calculus called it as (p,q)-generalization of Szász-Mirakjan operators.

Now we set the (p,q)-Szász-Mirakjan basis function as

$$s_n(p,q;x) =: E_{p,q}\big(-[n]_{p,q}x\big)\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{\big([n]_{p,q}x\big)^k}{[k]_{p,q}!}.$$

For $q \in (0,p)$, $p \in (q,1]$ and $x \in [0,\infty)$, $s_n(p,q;x) \geq 0$. We can easily check that

$$\sum_{k=0}^{\infty} s_n(p,q;x) =: E_{p,q}\big(-[n]_{p,q}x\big)\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{\big([n]_{p,q}x\big)^k}{[k]_{p,q}!} = 1.$$

For $0 < q < p \leq 1$ the (p,q)-Szász-Mirakjan operators are defined as

$$S_n(f,p,q;x) = [n]_{p,q} \sum_{k=0}^{\infty} p^{-k} q^k s_{n,k}(p,q;x) f\left(\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right), \quad x \in [0,\infty). \tag{7.3}$$

From the definition of the (p,q)-Szász-Mirakjan operators we derive the following formulas.

7.3.1 Recurrence relation

Lemma 7.5. *Let $0 < q < p \leq 1$. We have*

(i) $S_n(1,p,q;x) = 1;$

(ii) $S_n(t,p,q;x) = x;$

(iii) $S_n(t^2,p,q;x) = \frac{px^2}{q} + \frac{x}{[n]_{p,q}};$

(iv) $S_n(t^3,p,q;x) = \frac{p^3}{q^3}x^3 + \frac{p^2+2pq}{q[n]_{p,q}}x^2 + \frac{q^2}{[n]_{p,q}^2}x;$

(v) $S_n(t^4,p,q;x) = \frac{p^6}{q^6}x^4 + \frac{p^3(p^2+2q+3q^2)}{q^4[n]_{p,q}}x^3 + \frac{p(p^2+3pq+3q^2)}{q[n]_{p,q}}x^2 + \frac{q^3}{[n]_{p,q}^3}x.$

Now we propose our Kantorovich variant of (p,q)-Szász-Mirakjan operators (7.3) as follows:

For $f \in C[0,\infty)$, $0 < q < p \leq 1$ and each positive integer n,

$$K_n(f,p,q;x) = [n]_{p,q} \sum_{k=0}^{\infty} p^{-k} q^k s_{n,k}(p,q;x) \int_{\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{q^k[n]_{p,q}}} f(t) d_{p,q}t. \tag{7.4}$$

We will derive the recurrence formula for $K_n(t^m,p,q;x)$ and calculate $K_n(t^m,p,q;x)$ for $m = 0,1,2$.

Lemma 7.6. *For the operators K_n, we have*

$$K_n(t^m,p,q;x) = \frac{1}{[m+1]_{p,q}} \sum_{j=0}^{m} \sum_{i=0}^{j} \frac{p^i}{q^i [n]_{p,q}^{j-i}} \binom{j}{i} S_n(t^{m+i-j},p,q;x). \tag{7.5}$$

Proof. Using the expansion $a^{m+1} - b^{m+1} = (a-b)(a^m + a^{m-1}b + \cdots + ab^{m-1} + b^m)$ we have

$$\int_{\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{q^k[n]_{p,q}}} t^m d_{p,q}t = \frac{1}{[m+1]_{p,q}} \left\{ \left(\frac{[k+1]_{p,q}}{q^k[n]_{p,q}}\right)^{m+1} - \left(\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right)^{m+1} \right\}.$$

Using $[k+1]_{p,q} = p^k + q[k]_{p,q}$ and also $[k+1]_{p,q} = q^k + p[k]_{p,q}$, we have

$$\begin{aligned}
&\int_{\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{q^k[n]_{p,q}}} t^m d_{p,q}t \\
&= \frac{1}{[m+1]_{p,q}} \frac{p^k}{q^k[n]_{p,q}} \sum_{j=0}^{m} \left(\frac{[k+1]_{p,q}}{q^k[n]_{p,q}}\right)^j \left(\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right)^{m-j} \\
&= \frac{1}{[m+1]_{p,q}} \frac{p^k}{q^k[n]_{p,q}} \sum_{j=0}^{m} \left(\frac{q^k + p[k]_{p,q}}{q^k[n]_{p,q}}\right)^j \left(\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right)^{m-j} \\
&= \frac{1}{[m+1]_{p,q}} \frac{p^k}{q^k[n]_{p,q}} \sum_{j=0}^{m} \sum_{i=0}^{j} \binom{j}{i} \frac{p^i[k]_{p,q}^i q^{k(j-i)}}{q^{kj}[n]_{p,q}^j} \frac{[k]_{p,q}^{m-j}}{q^{(k-1)(m-j)}[n]_{p,q}^{m-j}} \\
&= \frac{1}{[m+1]_{p,q}} \frac{p^k}{q^k[n]_{p,q}} \sum_{j=0}^{m} \sum_{i=0}^{j} \binom{j}{i} \frac{p^i[k]_{p,q}^{m+i-j}}{q^{ki}[n]_{p,q}^m q^{(k-1)(m-j)}}.
\end{aligned}$$

Writing this in the definition of $K_n(t^m, p, q; x)$, we get

$$\begin{aligned}
&K_n(t^m, p, q; x) \\
&= [n]_{p,q} \sum_{k=0}^{\infty} p^{-k} q^k s_{n,k}(p, q; x) \int_{\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{q^k[n]_{p,q}}} t^m d_{p,q}t \\
&= \frac{1}{[m+1]_{p,q}} \sum_{j=0}^{m} \sum_{k=0}^{\infty} s_{n,k}(p, q; x) \sum_{i=0}^{j} \frac{p^i}{q^i[n]_{p,q}^{j-i}} \binom{j}{i} \frac{[k]_{p,q}^{m+i-j}}{q^{(k-1)(m+i-j)}[n]_{p,q}^{m+i-j}} \\
&= \frac{1}{[m+1]_{p,q}} \sum_{j=0}^{m} \sum_{i=0}^{j} \frac{p^i}{q^i[n]_{p,q}^{j-i}} \binom{j}{i} \sum_{k=0}^{\infty} \frac{[k]_{p,q}^{m+i-j}}{q^{(k-1)(m+i-j)}[n]_{p,q}^{m+i-j}} s_{n,k}(p, q; x) \\
&= \frac{1}{[m+1]_{p,q}} \sum_{j=0}^{m} \sum_{i=0}^{j} \frac{p^i}{q^i[n]_{p,q}^{j-i}} \binom{j}{i} S_n(t^{m+i-j}, p, q; x).
\end{aligned}$$

Using the recurrence formula (7.5), we may easily calculate $K_n(t^m, p, q; x)$ for $m = 0, 1, 2$. $\square$

Lemma 7.7. *We have*

(i) $K_n(1, p, q; x) = 1$;

(ii) $K_n(t, p, q; x) = \frac{1}{q}x + \frac{1}{[2]_{p,q}[n]_{p,q}}$;

(iii) $K_n(t^2, p, q; x) = \frac{p}{q^3}x^2 + \left(\frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}}\right)x + \frac{1}{[3]_{p,q}[n]_{p,q}^2}$;

(iv) $K_n(t^3,p,q;x) = \frac{p^3}{q^6}x^3+\left(\frac{p^2+2pq}{q^4[n]_{p,q}}+\frac{p(3p^2+2pq+q^2)}{q^3[4]_{p,q}[n]_{p,q}}\right)x^2+\left(\frac{1}{q[n]^2_{p,q}}+\frac{3p^2+2pq+q^2}{q^2[4]_{p,q}[n]^2_{p,q}}+\frac{3p+q}{q[4]_{p,q}[n]^2_{p,q}}\right)x+\frac{1}{[4]_{p,q}[n]^3_{p,q}}$;

(v) $K_n(t^4,p,q;x) = \frac{p^6}{q^{10}}x^4+\left(\frac{p^3(p^2+2q+3q^2)}{q^8[n]_{p,q}}+\frac{p^3(4p^3+3p^2q+2pq^2+q^3)}{q^6[5]_{p,q}[n]_{p,q}}\right)x^3+\left(\frac{p(p^2+3pq+3q^2)}{q^5[n]^2_{p,q}}+\frac{(p^2+2pq)(4p^3+3p^2q+2pq^2+q^3)}{q^4[5]_{p,q}[n]^2_{p,q}}+\frac{p(6p^2+3pq+q^2)}{q^3[5]_{p,q}[n]^2_{p,q}}\right)x^2+\left(\frac{1}{q[n]^3_{p,q}}+\frac{4p^3+3p^2q+2pq^2+q^3}{q[5]_{p,q}[n]^3_{p,q}}+\frac{6p^2+3pq+q^2}{q^2[5]_{p,q}[n]^3_{p,q}}+\frac{4p+q}{q[5]_{p,q}[n]^3_{p,q}}\right)x+\frac{1}{[5]_{p,q}[n]^4_{p,q}}$;

(vi) $K_n\big((t-x),p,q;x\big) = \frac{1-q}{q}x+\frac{1}{[2]_{p,q}[n]_{p,q}}$;

(vii) $K_n\big((t-x)^2,p,q;x\big) = \left(\frac{p}{q^3}-\frac{2}{q}+1\right)x^2+\left(\frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}}+\frac{1}{q^2[n]_{p,q}}-\frac{2}{[2]_{p,q}[n]_{p,q}}\right)x+\frac{1}{[3]_{p,q}[n]^2_{p,q}}$. (7.6)

Proof. Obviously, with the help of Lemma 7.5, we can get

$$\begin{aligned}
&K_n(t,p,q;x)\\
&=\frac{1}{[2]_{p,q}}\left\{\left(1+\frac{p}{q}\right)S_n(t,p,q;x)+\frac{1}{[n]_{p,q}}S_n(1,p,q;x)\right\}\\
&=\frac{1}{q}x+\frac{1}{[2]_{p,q}[n]_{p,q}},
\end{aligned}$$

$$\begin{aligned}
&K_n(t^2,p,q;x)\\
&=\frac{1}{[3]_{p,q}}\left\{\left(1+\frac{p}{q}+\frac{p^2}{q^2}\right)S_n(t^2,p,q;x)+\left(\frac{1}{[n]_{p,q}}+\frac{2p}{q[n]_{p,q}}\right)S_n(t,p,q;x)+\frac{1}{[n]^2_{p,q}}S_n(1,p,q;x)\right\}\\
&=\frac{1}{q^2}S_n(t^2,p,q;x)+\frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}}S_n(t,p,q;x)+\frac{1}{[3]_{p,q}[n]^2_{p,q}}S_n(1,p,q;x)\\
&=\frac{p}{q^3}x^2+\left(\frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}}+\frac{1}{q^2[n]_{p,q}}\right)x+\frac{1}{[3]_{p,q}[n]^2_{p,q}}.
\end{aligned}$$

Using the linearity of the operators, we can have

$$\begin{aligned}
&K_n\big((t-x)^2,p,q;x\big)\\
&=K_n(t^2,p,q;x)-2xK_n(t,p,q;x)+x^2K_n(1,p,q;x)\\
&=\frac{p}{q^3}x^2+\left(\frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}}+\frac{1}{q^2[n]_{p,q}}\right)x+\frac{1}{[3]_{p,q}[n]^2_{p,q}}-2x\left(\frac{1}{q}x+\frac{1}{[2]_{p,q}[n]_{p,q}}\right)+x^2\\
&=\left(\frac{p}{q^3}-\frac{2}{q}+1\right)x^2+\left(\frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}}+\frac{1}{q^2[n]_{p,q}}-\frac{2}{[2]_{p,q}[n]_{p,q}}\right)x+\frac{1}{[3]_{p,q}[n]^2_{p,q}}.
\end{aligned}$$

□

7.3.2 Convergence of operators

In this section we study the Korovkin's approximation property of the Kantorovich variant of (p,q)-Szász operators.

Theorem 7.8. *Let* $0 < q_n < p_n \leq 1$ *and* $A > 0$. *Then for each* $f \in C_m[0,\infty) := \{f \in C[0,\infty) : |f(x)| \leq M_f(1+x^m)$, *for some* $M_f > 0$ *depending on* f, $m > 0\}$ *where* $C_m[0,\infty)$ *be endowed with the norm* $\|f\|_m = \sup\limits_{x\in[0,\infty)} \frac{|f(x)|}{1+x^m}$, *the sequence of operators* $K_n(f,p_n,q_n;x)$ *converges to* f *uniformly on* $[0,A]$ *if and only if* $\lim_{n\to\infty} p_n = 1$ *and* $\lim_{n\to\infty} q_n = 1$.

Proof. First, we assume that $\lim_{n\to\infty} p_n = 1$ and $\lim_{n\to\infty} q_n = 1$. Now, we have to show that $K_n(f,p_n,q_n;x)$ converges to f uniformly on $[0,A]$.
From Lemma 7.7, we see that

$$K_n(1,p_n,q_n;x) \to 1, \quad K_n(t,p_n,q_n;x) \to x, \quad K_n(t^2,p_n,q_n;x) \to x^2$$

uniformly on $[0,A]$ as $n \to \infty$. Therefore, the well-known property of the Korovkin theorem implies that $K_n(f,p_n,q_n;x)$ converges to f uniformly on $[0,A]$ provided $f \in C_m[0,\infty)$.

We show the converse part by contradiction. Assume that p_n and q_n do not converge to 1. Then they must contain subsequences $p_{n_k} \in (0,1)$, $q_{n_k} \in (0,1)$, $p_{n_k} \to a \in [0,1)$ and $q_{n_k} \to b \in [0,1)$ as $k \to \infty$, respectively.
Thus,

$$\frac{1}{[n_k]_{p_{n_k},q_{n_k}}} = \frac{p_{n_k} - q_{n_k}}{(p_{n_k})^{n_k} - (q_{n_k})^{n_k}} \to 0 \text{ as } k \to \infty$$

and we get

$$K_n(t,p_{n_k},q_{n_k};x) - x = \frac{1}{q_{n_k}}x + \frac{1}{[2]_{p_{n_k},q_{n_k}}[n]_{p_{n_k},q_{n_k}}} - x \to \frac{x}{b} - x \neq 0.$$

This leads to a contradiction. Thus $p_n \to 1$ and $q_n \to 1$ as $n \to \infty$. □

7.3.3 Rate of convergence

Theorem 7.9. *Let* $f \in C_2[0,\infty)$, $q = q_n \in (0,1)$ *and* $p = p_n \in (q,1]$ *such that* $p_n \to 1$, $q_n \to 1$ *as* $n \to \infty$ *and* $\omega_{a+1}(f,\delta)$ *be the modulus of continuity on the finite interval*

$[0, a+1] \subset [0,\infty)$, *where* $a > 0$. *Then*

$$|K_n(f,p,q;x) - f(x)| \leq 4M_f(1+a^2)\delta_n^2(x) + 2\omega_{a+1}(f,\delta_n(x)),$$

where $\delta_n(x) = \sqrt{K_n\big((t-x)^2,p,q;x\big)}$, *given by (7.6).*

Proof. For $x \in [0,a]$ and $t > a+1$, since $t - x > 1$, we have

$$\begin{aligned}
|f(t) - f(x)| &\leq M_f(2 + x^2 + t^2) \\
&\leq M_f\big(2 + 3x^2 + 2(t-x)^2\big) \leq M_f(4+3x^2)(t-x)^2 \leq 4M_f(1+a^2)(t-x)^2. \quad (7.7)
\end{aligned}$$

For $x \in [0,a]$ and $t \leq a+1$, we have

$$|f(t) - f(x)| \leq \omega_{a+1}(f,|t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta) \quad (7.8)$$

with $\delta > 0$. From (7.7) and (7.8), we may write

$$|f(t) - f(x)| \leq 4M_f(1+a^2)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta), \quad (7.9)$$

for $x \in [0,a]$ and $t \geq 0$. Thus, by applying the Cauchy-Schwartz's inequality, we have

$$\begin{aligned}
&|K_n(f,p,q;x) - f(x)| \\
&\leq K_n\big(|f(t) - f(x)|,p,q;x\big) \\
&\leq 4M_f(1+a^2)K_n\big((t-x)^2,p,q;x\big) + \left(1 + \frac{1}{\delta}\sqrt{K_n\big((t-x)^2,p,q;x\big)}\right)\omega_{a+1}(f,\delta) \\
&\leq 4M_f(1+a^2)\delta_n^2(x) + 2\omega_{a+1}(f,\delta_n(x))
\end{aligned}$$

on choosing $\delta := \delta_n(x)$. This completes the proof of the theorem. □

7.3.4 Weighted approximation

Let $f \in C_2^*[0,\infty) := \left\{f \in C_2[0,\infty) : \lim\limits_{x\to\infty} \frac{|f(x)|}{1+x^2} < \infty\right\}$. Throughout the section, we assume that (p_n) and (q_n) are sequences such that $0 < q_n < p_n \leq 1$ and $p_n \to 1$, $q_n \to 1$ as $n \to \infty$.

Theorem 7.10. *For each* $f \in C_2^*[0,\infty)$, *we have*

$$\lim_{n\to\infty} \|K_n(f,p_n,q_n;x) - f(x)\|_2 = 0.$$

Proof. Using the Korovkin type theorem on weighted approximation in [30], we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \|K_n(t^i, p_n, q_n; x) - x^i\|_2 = 0, \quad i = 0, 1, 2. \tag{7.10}$$

Since $K_n(1, p_n, q_n; x) = 1$, (7.10) holds true for $i = 0$.

By Lemma 7.7, we have

$$\begin{aligned}
&\|K_n(t, p_n, q_n; x) - x\|_2 \\
&= \sup_{x\in[0,\infty)} \frac{|K_n(t, p_n, q_n; x) - x|}{1+x^2} \\
&= \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|\frac{1}{q_n}x + \frac{1}{[2]_{p,q}[n]_{p,q}} - x\right| \\
&\leq \left(\frac{1}{q_n} - 1\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{1}{[2]_{p,q}[n]_{p,q}} \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\
&\leq \frac{1}{q_n} - 1 + \frac{1}{[2]_{p,q}[n]_{p,q}},
\end{aligned}$$

which implies that the condition in (7.10) holds for $i = 1$ as $n \to \infty$.

Similarly we can write

$$\begin{aligned}
&\|K_n(t^2, p_n, q_n; x) - x^2\|_2 \\
&= \sup_{x\in[0,\infty)} \frac{|K_n(t^2, p_n, q_n; x) - x^2|}{1+x^2} \\
&\leq \left(\frac{p_n}{q_n^3} - 1\right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} \\
&+ \left(\frac{2p_n + q_n}{q_n[3]_{p_n,q_n}[n]_{p_n,q_n}} + \frac{1}{q_n^2[n]_{p_n,q_n}}\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\
&+ \frac{1}{[3]_{p_n,q_n}[n]^2_{p_n,q_n}} \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\
&\leq \frac{p_n}{q_n^3} - 1 + \frac{2p_n + q_n}{q_n[3]_{p_n,q_n}[n]_{p_n,q_n}} + \frac{1}{q_n^2[n]_{p_n,q_n}} + \frac{1}{[3]_{p_n,q_n}[n]^2_{p_n,q_n}},
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|K_n(t^2, p_n, q_n; x) - x^2\|_2 = 0,$$

(7.10) holds for $i = 2$. Thus the proof is completed. □

We give the following theorem to approximate all functions in $C_2^*[0, \infty)$. This type of results are given in [27] for classical Szász operators.

Theorem 7.11. *For each $f \in C_2^*[0,\infty)$ and $\alpha > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\left|K_n(f,p_n,q_n;x) - f(x)\right|}{(1+x^2)^{1+\alpha}} = 0.$$

Proof. Let $x_0 \in [0,\infty)$ be arbitrary but fixed. Then

$$\begin{aligned}
&\sup_{x\in[0,\infty)} \frac{\left|K_n(f,p_n,q_n;x) - f(x)\right|}{(1+x^2)^{1+\alpha}} \\
&= \sup_{x\leq x_0} \frac{\left|K_n(f,p_n,q_n;x) - f(x)\right|}{(1+x^2)^{1+\alpha}} + \sup_{x>x_0} \frac{\left|K_n(f,p_n,q_n;x) - f(x)\right|}{(1+x^2)^{1+\alpha}} \\
&\leq \|K_n(f) - f\|_{C[0,x_0]} + \|f\|_2 \sup_{x>x_0} \frac{\left|K_n(1+t^2,p,q;x)\right|}{(1+x^2)^{1+\alpha}} + \sup_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha}}. \qquad (7.11)
\end{aligned}$$

Since $|f(x)| \leq \|f\|_2(1+x^2)$, we have $\sup\limits_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha}} \leq \frac{\|f\|_2}{(1+x_0^2)^\alpha}$.

Let $\varepsilon > 0$ be arbitrary. We can choose x_0 to be so large that

$$\frac{\|f\|_2}{(1+x_0^2)^\alpha} < \frac{\varepsilon}{3}. \qquad (7.12)$$

In view of Theorem 7.8, we obtain

$$\begin{aligned}
&\|f\|_2 \lim_{n\to\infty} \frac{\left|K_n(1+t^2,p,q;x)\right|}{(1+x^2)^{1+\alpha}} \\
&= \frac{1+x^2}{(1+x^2)^{1+\alpha}}\|f\|_2 = \frac{\|f\|_2}{(1+x^2)^\alpha} \leq \frac{\|f\|_2}{(1+x_0^2)^\alpha} < \frac{\varepsilon}{3}. \qquad (7.13)
\end{aligned}$$

Using Theorem 7.11, we can see that the first term of the inequality (7.11), implies that

$$\|K_n(f) - f\|_{C[0,x_0]} < \frac{\varepsilon}{3}, \quad \text{as } n \to \infty. \qquad (7.14)$$

Combining (7.12)-(7.14), we get that desired result. □

For $f \in C_2^*[0,\infty)$, the weighted modulus of continuity is defined as

$$\Omega_2(f,\delta) = \sup_{x\geq 0, 0<h\leq\delta} \frac{|f(x+h) - f(x)|}{1+(x+h)^2}.$$

Lemma 7.12. *([64]) If $f \in C_2^*[0,\infty)$ then*

(i) $\Omega_2(f,\delta)$ *is monotone increasing function of* δ,

(ii) $\lim\limits_{\delta\to 0+} \Omega_2(f,\delta) = 0$,

(iii) *for any* $\lambda \in [0,\infty)$, $\Omega_2(f,\lambda\delta) \leq (1+\lambda)\Omega_2(f,\delta)$.

Theorem 7.13. *If* $f \in C_2^*[0,\infty)$, *then for sufficiently large* n *we have*

$$\left|K_n(f,p,q;x) - f(x)\right| \leq K(1+x^{2+\lambda})\Omega_2(f,\delta_n), \quad x \in [0,\infty),$$

where $\lambda \geq 1$, $\delta_n = \max\{\alpha_n, \beta_n, \gamma_n\}$, $\alpha_n, \beta_n, \gamma_n$ *being*

$$\alpha_n = \frac{p}{q^3} - \frac{2}{q} + 1, \quad \beta_n = \frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}} - \frac{2}{[2]_{p,q}[n]_{p,q}}, \quad \gamma_n = \frac{1}{[3]_{p,q}[n]_{p,q}^2}.$$

Proof. From the definition of $\Omega_2(f,\delta)$ and Lemma 7.12, we may write

$$\begin{aligned}
&|f(t) - f(x)| \\
&\leq \left(1+(x+|t-x|)^m\right)\left(\frac{|t-x|}{\delta}+1\right)\Omega_2(f,\delta) \\
&\leq \left(1+(2x+t)^m\right)\left(\frac{|t-x|}{\delta}+1\right)\Omega_2(f,\delta) \\
&:= \varphi_x(t)\left(1+\frac{1}{\delta}\psi_x(t)\right)\Omega_2(f,\delta).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&|K_n(f,p,q;x) - f(x)| \\
&\leq \Omega_2(f,\delta_n)\left(K_n(\varphi_x,p,q;x) + \frac{1}{\delta_n}K_n\left(\varphi_x\psi_x,p,q;x\right)\right).
\end{aligned}$$

Applying the Cauchy-Schwartz inequality to the second term on the right-hand side, we get

$$\begin{aligned}
&|K_n(f,p,q;x) - f(x)| \\
&\leq \Omega_2(f,\delta_n)\Big(K_n(\varphi_x,p,q;x) + \frac{1}{\delta_n}\sqrt{K_n(\varphi_x^2,p,q;x)}\sqrt{K_n(\psi_x^2,p,q;x)}\Big).
\end{aligned} \tag{7.15}$$

From Lemma 7.7, we get

$$\begin{aligned}
&\frac{1}{1+x^2}K_n(1+t^2,p,q;x) \\
&= \frac{1}{1+x^2} + \frac{p}{q^3}\frac{x^2}{1+x^2} + \left(\frac{p+[2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}}\right)\frac{x}{1+x^2} + \frac{1}{[3]_{p,q}[n]_{p,q}^2}\frac{1}{1+x^2} \\
&\leq 1 + C_1, \text{ for sufficiently large } n
\end{aligned} \tag{7.16}$$

where C_1 is a positive constant. From (7.16), there exists a positive constant K_1 such that $K_n(\varphi_x, p, q; x) \leq K_1(1+x^2)$, for sufficiently large n.

Proceeding similarly $\frac{1}{1+x^4}K_n(1+t^4, p, q; x) \leq 1 + C_2$, for sufficiently large n, where C_2 is a positive constant. So there exists a positive constant K_2 such that $K_n(\varphi_x^2, p, q; x) \leq K_2(1+x^2)$, where $x \in [0, \infty)$ and n is large enough. Also we get

$$\begin{aligned}
&K_n(\psi_x^2, p, q; x) \\
&= \left(\frac{p}{q^3} - \frac{2}{q} + 1\right)x^2 + \left(\frac{p + [2]_{p,q}}{q[3]_{p,q}[n]_{p,q}} + \frac{1}{q^2[n]_{p,q}} - \frac{2}{[2]_{p,q}[n]_{p,q}}\right)x + \frac{1}{[3]_{p,q}[n]_{p,q}^2} \\
&= \alpha_n x^2 + \beta_n x + \gamma_n.
\end{aligned}$$

Hence form (7.15), we have

$$|K_n(f, p, q; x) - f(x)| \leq (1+x^2)\Big(K_1 + \frac{1}{\delta_n}K_2\sqrt{\alpha_n x^2 + \beta_n x + \gamma_n}\Big)\Omega_2(f, \delta_n).$$

If we take $\delta_n = \max\{\alpha_n, \beta_n, \gamma_n\}$, then we get

$$\begin{aligned}
&|K_n(f, p, q; x) - f(x)| \\
&\leq (1+x^2)\Big(K_1 + K_2\sqrt{x^2 + x + 1}\Big)\Omega_2(f, \delta_n) \\
&\leq K_3(1 + x^{2+\lambda})\Omega_2(f, \delta_n), \quad \text{for sufficiently large } n \text{ and } x \in [0, \infty).
\end{aligned}$$

Hence the proof is completed. $\square$

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On the stability of some positive linear operators from approximation theory

M. Mursaleen · Khursheed J. Ansari

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Abstract Recently, Popa and Raşa have shown the stability/ instability of some classical operators defined on [0, 1] and obtained the best constant when the positive linear operators are stable in the sense of Hyers–Ulam. In this paper we show that the Kantorovich–Stancu type operators, King's operator, Bernstein–Stancu type operators, and Kantorovich–Bernstein–Stancu type operators with shifted knots are Hyers–Ulam stable. Further we find the best Hyers–Ulam stability constants for some of these operators. We also prove that Szász–Mirakjan and Kantorovich–Szász–Mirakjan type operators are unstable in the sense of Hyers and Ulam.

Keywords Hyers–Ulam stability · Kantorovich–Stancu type operator · King's operator · Kantorovich–Bernstein–Stancu type operator · Szász–Mirakjan type operator · Kantorovich–Szász–Mirakjan type operator · Best constant

Mathematics Subject Classification Primary 39B82; Secondary 41A35 · 41A44

1 Introduction

The equation of homomorphism is stable if every "approximate" solution can be approximated by a solution of this equation. The problem of stability of a functional equation was formulated by Ulam [1] in a conference at Wisconsin University, Madison

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Approximation of q-Stancu-Beta Operators Which Preserve x^2

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Abstract In this paper, we study some approximation properties of q-analogue of Stancu-Beta operators which preserve x^2. We determine the rate of global convergence in weighted spaces. We also prove the Voronovskaja-type theorem for these operators.

Keywords q-Analogue of Stancu-Beta operators · Modulus of continuity · Voronovskaja-type theorem · Korovkin-type approximation theorem

Mathematics Subject Classification 41A10 · 41A25 · 41A36

1 Introduction and Preliminaries

It has been observed that most of the approximating operators, L_n, preserve $e_i(x) = x^i$ $(i = 0, 1)$, i.e. $L_n(e_0; x) = e_0(x)$ and $L_n(e_1; x) = e_1(x)$, $n \in \mathbb{N}$. These conditions hold specially, for the Bernstein polynomials, Szász–Mirakjan operators, and the Baskakov operators (see [2, 11]). For each of these operators, $L_n(e_2; x) \neq e_2(x) = x^2$. King [9] has presented a non-trivial sequence $\{V_n\}$ of positive linear operators which approximate each continuous function on $[0, 1]$ while preserving the functions e_0 and e_2, i.e. $V_n : C[0, 1] \to C[0, 1]$, for any $n \in \mathbb{N}$ and for any function $f \in C[0, 1]$, is given as follows

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On (p, q)-analogue of Bernstein operators

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ABSTRACT

In this paper, we introduce a new analogue of Bernstein operators and we call it as (p, q)-Bernstein operators which is a generalization of q-Bernstein operators. We also study approximation properties based on Korovkin's type approximation theorem of (p, q)-Bernstein operators and establish some direct theorems. Furthermore, we show comparisons and some illustrative graphics for the convergence of operators to a function.

1. Introduction and preliminaries

In 1912, and Bernstein [6] introduced the following sequence of operators $B_n\colon C[0,1] \to C[0,1]$ defined for any $n \in \mathbb{N}$ and for any function $f \in C[0,1]$

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1]. \tag{1.1}$$

Later it was found that Bernstein polynomials possess many remarkable properties, so new applications and generalizations are being discovered of it. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design, and solutions of differential equations. The rapid development of q-calculus has led to the discovery of new generalizations of Bernstein polynomials involving q-integers. Lupaş [18] was the first who introduced the q-analogue of the well known Bernstein polynomials and investigated its approximating and shape-preserving properties. Let $f \in C[0,1]$. The linear operators $L_{n,q}\colon C[0,1] \to C[0,1]$, defined by

$$L_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) b^q_{n,k}(x), \tag{1.2}$$

where

$$b^q_{n,k}(x) = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k (1-x)^{n-k}}{\prod_{j=0}^{n-1}(1-x+q^j x)}$$

are called Lupaş q-analogue of Bernstein polynomials.

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Some approximation results by (p, q)-analogue of Bernstein–Stancu operators

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ABSTRACT

In this paper, we introduce a new analogue of Bernstein–Stancu operators based on (p, q)-integers which we call as (p, q)-Bernstein–Stancu operators. We study approximation properties for these operators based on Korovkin's type approximation theorem and also study some direct theorems. Furthermore, we give comparisons and some illustrative graphics for the convergence of operators to some function.

1. Introduction and preliminaries

During the last two decades, the applications of q-calculus emerged as a new area in the field of approximation theory. The rapid development of q-calculus has led to the discovery of various generalizations of Bernstein polynomials involving q-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations.

In 1987, Lupaş [17] introduced the first q-analogue of Bernstein operators [6] and investigated its approximating and shape-preserving properties. Another q-generalization of the classical Bernstein polynomials is due to Phillips [26]. Several generalizations of well-known positive linear operators based on q-integers were introduced and their approximation properties have been studied by several authors. For instance, q-Baskakov–Kantorovich operators in [11]; q -Szász–Mirakjan operators in [25]; q-Bleimann, Butzer and Hahn operators in [3] and [9]; q-analogue of Baskakov and Baskakov–Kantorovich operators in [18]; q-analogue of Szász–Kantorovich operators in [19]; q-analogue of Stancu-Beta operators in [4] and [21]; and q-Lagrange polynomials in [23] were defined and their approximation properties were investigated.

Recently, Mursaleen et al. introduced and studied approximation properties for new positive linear operators of Lagrange type in [22] and also studied approximation properties of the q-analogue of generalized Berstein–Shurer operators in [20].

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